The Principle of the Fermionic Projector

Chapters 0-4

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Abstract

The "principle of the fermionic projector" provides a new mathematical framework for the formulation of physical theories and is a promising approach for physics beyond the standard model. The book begins with a brief review of relativity, relativistic quantum mechanics and classical gauge theories, with the emphasis on the basic physical concepts and the mathematical foundations. The external field problem and Klein's paradox are discussed and then resolved by introducing the so-called fermionic projector, a global object in space-time which generalizes the notion of the Dirac sea. The mathematical core of the book is to give a precise definition of the fermionic projector and to employ methods of hyperbolic differential equations for its detailed analysis. The fermionic projector makes it possible to formulate a new type of variational principles in space-time. The mathematical tools for the analysis of the corresponding Euler-Lagrange equations are developed. A particular variational principle is proposed which gives rise to an effective interaction showing many similarities to the interactions of the standard model.

The main chapters of the book are easily accessible for beginning graduate students in mathematics or physics. Several appendices provide supplementary material which will be useful to the experienced researcher.

Contents

Preface

The basic ideas behind the "principle of the fermionic projector" go back to the years 1990-91 when I was a physics student in Heidelberg. At that time, I was excited about relativity and quantum mechanics, in particular about classical Dirac theory, but I felt uncomfortable with quantum field theory. The dissatisfaction with second quantized fields, which was in part simply a beginner's natural skepticism towards an unfamiliar physical concept, was my motivation for thinking about possible alternatives. Today I clearly understand quantum field theory much better, and many of my early difficulties have disappeared. Nevertheless, some of my objections remain, and the idea of formulating physics in a unified way based on Dirac's original concept of a "sea of interacting particles" seems so natural and promising to me that I have pursued this idea ever since. It took quite long to get from the first ideas to a consistent theory, mainly because mathematical methods had to be developed in order to understand the "collective behavior" of the particles of the Dirac sea.

This book gives me the opportunity to present the main ideas and methods in a somewhat broader context, with the intention of making this area of mathematical physics accessible to both theoretical physicists and applied mathematicians. The emphasis of the main chapters is on the conceptual part, whereas the more technical aspects are worked out in the appendices.

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Felix Finster, Regensburg, November 2004

Preface to the Online Edition

In the few years since the book appeared, I was frequently asked if the introductory chapters were also available online. Also, I heard complaints that the preprints on the arXiv on the "principle of the fermionic projector" were preliminary versions which were not quite compatible with the book and with subsequent papers. In order to improve the situation, I decided to replace my original preprints hep-th/0001048, hep-th/0202059 and hep-th/0210121 by the corresponding chapters of the book.

I took the opportunity to correct some typos. I also added a few footnotes beginning with "Online version:" which point out to later developments. Apart from these changes, the present online version coincides precisely with the book in the AMS/IP series. In particular, all equation numbers are the same.

Felix Finster, Regensburg, October 2009

CHAPTER 0

The Principle of the Fermionic Projector – A New Mathematical Model of Space-Time

The mathematical model of space-time has evolved in history. In Newtonian mechanics, space is described by a Euclidean vector space. In special relativity, space and time were combined to Minkowski space, a vector space endowed with a scalar product of signature $(+---)$. In general relativity, the vector space structure of space-time was given up on the large scale and was replaced by that of a Lorentzian manifold. The first hint that the notions of space and time should be modified also on the microscopic scale was obtained by Planck, who observed that the gravitational constant, Planck's constant and the speed of light give rise to a quantity of the dimension of length,

$$l_P \;=\; \sqrt{\frac{\hbar\,\kappa}{c^3}} \;\approx\; 1.6 \cdot 10^{-35}\,\text{m}\,,$$

and he conjectured that for distances as tiny as this so-called Planck length, the conventional laws of physics should no longer hold, and yet unknown physical effects might become significant. Later, this picture was confirmed by quantum field theory. Namely, due to the ultraviolet divergences, perturbative QFT is well-defined only after regularization, and the regularization is then removed using the renormalization procedure. While renormalization ensures that the observable quantities do not depend on the regularization, the theoretical justification for the renormalization program lies in the assumption that the continuum theory should be valid only down to some microscopic length scale, and it seems natural to associate this length scale to the Planck length.

Today most physicists agree that in order to make progress in fundamental physics, one should go beyond the continuum field theory and try to get a better understanding of the microscopic structure of space-time. However, giving up the usual space-time continuum leads to serious problems, and this is one reason why there is no consensus on what the correct mathematical model for "Planck scale physics" should be. Let us illustrate the difficulties by briefly discussing a few of the many approaches. The simplest and maybe most natural approach is to assume that on the Planck scale space-time is no longer a continuum but becomes in some way "discrete." This idea is for example used in lattice gauge theories, where space-time is modeled by a four-dimensional lattice (see Figure 0.1(a)). Using the specific structures of a lattice like the nearest-neighbor relation and the lattice spacing d, one can set up a quantum field theory which is ultraviolet finite [**Ro**]. Lattice gauge theories are very useful for numerical simulations [**K**]. However, they are not fully satisfying from a conceptual point of view because a space-time lattice is not consistent with the equivalence principle of general relativity. Namely, if one considers the lattice in the reference frame of an accelerated observer (denoted in in Figure 0.1(b) by (t', x')), the lattice points are no longer in a regular configuration. Thus the structure of a lattice is not invariant under

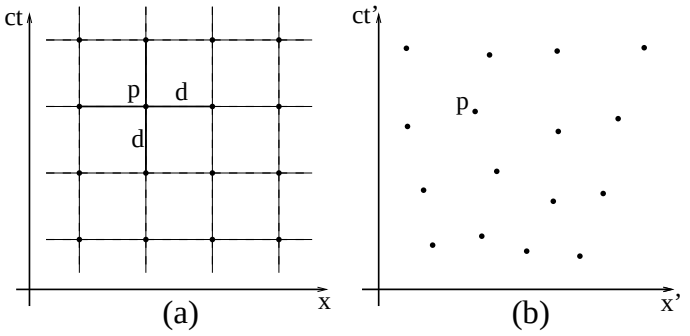

FIGURE 0.1. A lattice regularization for two different observers.

general coordinate transformations and hence is not compatible with the equivalence principle.

An alternative approach is to hold on to a space-time continuum, but to work with objects which are spread out in space-time, thereby avoiding the ultraviolet problems which occur for very small distances. The most prominent example in this direction is string theory, where physics is on a fundamental level described by so-called strings, which are extended in space-time and are therefore ultraviolet finite. The basic problem with such theories is that they are formulated using the structures of an underlying continuum space-time (like the vector space structure, the topology or even a metric), although all observable quantities (like the Lorentz metric, particles, fields, etc.) are to be derived from the non-localized objects, without referring to the underlying space-time continuum. Therefore, the structures of the underlying "continuum background" may seem artificial, and serious conceptual problems arise when these background structures are not compatible with basic physical principles (for example, a background vector space is not compatible with the equivalence principle). For short, one says that the theory is not background-free (for a more detailed discussion see [**Ba**] and the references therein).

Thus one difficulty in finding a promising model for Planck scale physics is that it should be background-free and should respect the basic physical principles (like the equivalence principle, the local gauge principle, etc.). There are approaches which actually meet these requirements. One is Connes' noncommutative geometry. As pointed out by Grothendieck, there is a one-to-one correspondence between the points of a manifold and the prime ideals of the (commutative) algebra of functions on this manifold. Thus the geometry of a manifold can be recovered from an underlying algebraic structure, and this makes it possible to extend the notions of space and time by considering more general, noncommutative algebras (see [**Co, CC**] for details and physical applications). The other approach is quantum gravity as pursued by Ashtekar and his school [**ARS, Th**]. The hope is that the ultraviolet divergences of QFT should disappear as soon as gravity, quantized in a non-perturbative way, is included.

Ultimately, a model for space-time on the Planck scale must be verified or falsified by physical experiments. Unfortunately, experiments on the Planck scale would require such enormously high energies that they are at present out of reach. Indirect experiments seem possible in principle [**ARA**] but have so far not been carried out. In my opinion, one should not hope for important new experimental input in the near future, but one should try to make due with the experimental data which is now available.

Indeed, this situation is not as hopeless as it might appear at first sight. Namely, in present physical models like the standard model, a lot of information from experiments is built in empirically, like the masses of the elementary particles, the gauge groups, the coupling constants, etc. Therefore, one can get a connection to experiments simply by trying to reproduce this well known empirical data. If successful, this procedure could give strong physical evidence for a model. For example, even if based on ad-hoc assumptions on the microscopic structure of space-time (which cannot be verified directly), a model would be very convincing and physically interesting if it gave the correct value for the fine structure constant and explained e.g. why the strong gauge group is $SU(3)$ or why neutrinos do not couple to the electromagnetic field. Thus the goal of a mathematical model for space-time on the Planck scale is to give a more fundamental explanation for the structures and empirical parameters in the standard model. To our opinion, only such concrete results can justify the model. Clearly, it is far from obvious how a promising model should look, or even in which mathematical framework it should be formulated. But at least for a mathematician, this makes the problem only more interesting, and it seems a challenging program to search for such models and to develop the mathematical methods needed for their analysis.

Our point of view that the mathematical model needs justification by known experimental data is not just a requirement which the model should fulfill at the very end, but it also gives a few hints on how one should proceed in order to find a promising model. First of all, one can expect concrete results only if one makes specific assumptions. Therefore, generalizing the notion of a Lorentzian manifold does not seem to be sufficient, but one should make a concrete ansatz for the microscopic structure of space-time (as it is done e.g. in string theory and lattice gauge theories). Furthermore, in order to make it easier to get a connection to well-established theories like classical field theory and quantum mechanics, it seems a good idea to take these theories as the starting point and to try to work as closely as possible with the objects used in these theories. Namely, if one drops important structures of the classical theories and/or introduces too many new structures ad hoc, it might become very difficult if not impossible to obtain a relation to observable data.

In our model of space-time we have tried to follow the above considerations. Our starting point is relativistic quantum mechanics and classical field theory. We assume that space-time is discrete on the Planck scale. But our notion of "discrete space-time" is much more general than a four-dimensional lattice; in particular, we do not assume any discrete symmetries in space-time, we keep the local gauge freedom, and we also extend the diffeomorphism invariance of a manifold in such a way that the equivalence principle is respected in discrete space-time. Furthermore, our model is background-free. In contrast to string theory, we do not introduce any new objects, but hold on to the structures already present in classical Dirac theory. We build in our physical ideas simply by prescribing which of these structures we consider as being fundamental, and then carry over these structures to discrete space-time. In the resulting mathematical framework, it is impossible to formulate the conventional physical equations, and thus we propose instead new equations of different type, referred to as the equations of discrete space-time. In a certain limiting case, the so-called continuum limit, we get a connection to the conventional formulation of physics in a space-time continuum. We point out that, in contrast to the Ashtekar program, we do not work with second quantized fields. But our concept is that the equations of discrete space-time should

also account for the physical effects of quantized fields if one goes beyond the continuum limit.

More specifically, we describe the physical system by the *fermionic projector* $P(x,y)$, which can be regarded as the projector on all occupied fermionic states of the system, including the states of the Dirac sea. After carrying over the fermionic projector to discrete space-time, we can set up variational principles like our "model variational principle"

$$\sum_{x,y \in M} \mathcal{L}[P(x,y)\, P(y,x)] \;\rightarrow\; \min ,$$

where the "Lagrangian" $\mathcal{L}$ is given by

$$\mathcal{L}[A] \;=\; |A^2| - \mu |A|^2 ,$$

with μ a Lagrangian multiplier. Here $|A|$ is the so-called spectral weight defined as the sum of the absolute values of the eigenvalues of the matrix A (or, in case that A is not diagonalizable, of the zeros of its characteristic polynomial). We study the above variational principle for a fermionic projector which in the vacuum is the direct sum of seven identical massive sectors and one massless left-handed sector, each of which is composed of three Dirac seas. Analyzing the continuum limit for an interaction via general chiral and (pseudo)scalar potentials, we find that the sectors spontaneously form pairs, which are referred to as blocks. The resulting so-called effective interaction can be described by chiral potentials corresponding to the effective gauge group

$$SU(2) \times SU(3) \times U(1)^3 .$$

This model has striking similarity to the standard model if the block containing the left-handed sector is identified with the leptons and the three other blocks with the quarks. Namely, the effective gauge fields have the following properties.

- The $SU(3)$ corresponds to an unbroken gauge symmetry. The $SU(3)$ gauge fields couple to the quarks exactly as the strong gauge fields in the standard model.
- The $SU(2)$ potentials are left-handed and couple to the leptons and quarks exactly as the weak gauge potentials in the standard model. Similar to the CKM mixing in the standard model, the off-diagonal components of these potentials must involve a non-trivial mixing of the generations. The $SU(2)$ gauge symmetry is spontaneously broken.
- The $U(1)$ of electrodynamics can be identified with an Abelian subgroup of the effective gauge group.

The effective gauge group is larger than the gauge group of the standard model, but this is not inconsistent because a more detailed analysis of our variational principle should give further constraints for the Abelian gauge potentials. Moreover, there are the following differences to the standard model, which we derive mathematically without working out their physical implications.

- The $SU(2)$ gauge field tensor F must be simple in the sense that $F = \Lambda\, s$ for a real 2-form Λ and an $su(2)$-valued function s.
- In the lepton block, the off-diagonal $SU(2)$ gauge potentials are associated with a new type of potential, called nil potential, which couples to the right-handed component.

These results give a strong indication that the principle of the fermionic projector is of physical significance. Nevertheless, the goal of this book is not to work out our model

variational principle in all details. Our intention is to develop the general concepts and methods from the basics, making them easily accessible to the reader who might be interested in applying them to other equations of discrete space-time or to related problems.

These notes are organized as follows. In order to make the presentation self-contained, Chapter 1 gives a brief account of the mathematical and physical preliminaries. Chapter 2 introduces the fermionic projector in the continuum and provides the mathematical methods needed for its detailed analysis. In Chapter 3 we go beyond the continuum description and introduce our mathematical model for space-time on the Planck scale. In Chapter 4 we develop a mathematical formalism suitable for the analysis of the continuum limit. In Chapter 5 we present and discuss different equations of discrete space-time in the vacuum, and we choose the most promising equations as our "model equations". In the last Chapters 6-8 we analyze interacting systems in the continuum limit. The appendices contain additional material and will be referred to from the main chapters.

CHAPTER 1

Preliminaries

1.1. Relativity

In this section we briefly outline the mathematical framework of special and general relativity (for a more detailed introduction see [**Wo**] and [**Wa**]). We always work in *normal units* where $\hbar = c = 1$. In special relativity, space-time is described by Minkowski space $(M, \langle .,. \rangle)$, a real 4-dimensional vector space endowed with an inner product of signature $(+---)$. Thus, choosing a pseudo-orthonormal basis $(e_i)_{i=0,\ldots,3}$ and representing the vectors of M in this basis, $\xi = \sum_{i=0}^3 \xi^i e_i$, the inner product takes the form

$$\langle \xi, \eta \rangle \;=\; \sum_{j,k=0}^{3} g_{jk}\, \xi^j\, \eta^k \,, \tag{1.1.1}$$

where g_{ij}, the *Minkowski metric*, is the diagonal matrix $g = \text{diag}\,(1, -1, -1, -1)$. In what follows we usually omit the sums using Einstein's summation convention (i.e. we sum over all indices which appear twice, once as an upper and once as a lower index). Also, we sometimes abbreviate the Minkowski scalar product by writing $\xi\eta := \langle \xi, \eta \rangle$ and $\xi^2 := \langle \xi, \xi \rangle$. A pseudo-orthonormal basis $(e_i)_{i=0,\ldots,3}$ is in physics called a *reference frame*, because the corresponding coordinate system (x^i) of Minkowski space gives the time and space coordinates for an observer in a system of inertia. We also refer to $t := x^0$ as time and denote the spatial coordinates by $\vec{x} = (x^1, x^2, x^3)$.

The sign of the Minkowski metric encodes the causal structure of space-time. Namely, a vector $\xi \in M$ is said to be

$$\left.\begin{array}{cc} \textit{timelike} & \text{if } \langle \xi, \xi \rangle > 0 \\ \textit{spacelike} & \text{if } \langle \xi, \xi \rangle < 0 \\ \textit{null} & \text{if } \langle \xi, \xi \rangle = 0\,. \end{array}\right\}$$

Likewise, a vector is called *non-spacelike* if it is timelike or null. The null vectors form the double cone $L = \{\xi \in M \,|\, \langle \xi, \xi \rangle = 0\}$, referred to as the *light cone*. Physically, the light cone is formed of all rays through the origin of M which propagate with the speed of light. Similarly, the timelike vectors correspond to velocities slower than light speed; they form the *interior light cone* $I = \{\xi \in M \,|\, \langle \xi, \xi \rangle > 0\}$. Finally, we introduce the *closed light cone* $J = \{\xi \in M \,|\, \langle \xi, \xi \rangle \geq 0\}$. The space-time trajectory of a moving object describes a curve $q(\tau)$ in Minkowski space (with τ an arbitrary parameter). We say that the space-time curve q is timelike if the tangent vector to q is everywhere timelike. Spacelike, null, and non-spacelike curves are defined analogously. The usual statement of causality that no information can travel faster than with the speed of light can then be expressed as follows,

causality: information can be transmitted only along non-spacelike curves.

The set of all points which can be joined with a given space-time point x by a non-spacelike curve is precisely the closed light cone centered at x, denoted by $J_x := J - x$. It is the union of the two single cones

$$\begin{aligned} J_x^\vee &= \{y \in M \,|\, (y-x)^2 \geq 0,\ (y^0 - x^0) \geq 0\} \\ J_x^\wedge &= \{y \in M \,|\, (y-x)^2 \geq 0,\ (y^0 - x^0) \leq 0\}\,, \end{aligned}$$

which have the interpretation as the points in the causal future and past of x, respectively. Thus we refer to $J_x^\vee$ and $J_x^\wedge$ as the closed *future* and *past light cones* centered at x, respectively. Similarly, we also introduce the sets $I_x^\vee$, $I_x^\wedge$ and $L_x^\vee$, $L_x^\wedge$.

A linear transformation of Minkowski space which leaves the form of the Minkowski metric (1.1.1) invariant is called a *Lorentz transformation*. The Lorentz transformations form a group, the *Lorentz group*. The Lorentz transformations which preserve both the time direction and the space orientation form a subgroup of the Lorentz group, the *orthochronous proper Lorentz group*.

The physical equations should be *Lorentz invariant*, meaning that they must be formulated in Minkowski space, independent of the reference frame. A convenient way of making Lorentz invariance manifest is to bring the equations in tensorial form (see [**L**] for a good introduction). Writing out the tensor indices, we get upper "contravariant" and lower "covariant" indices, which can be transformed into each other by contracting with the metric, e.g. $\xi_j = g_{jk}\xi^k$ and $\xi^k = g^{kl}\xi_l$ with $g^{kl} = (g_{kl})^{-1}$. In order to formulate electrodynamics in a manifestly Lorentz invariant form, one combines the electric potential and the vector potential to a 1-form $A = A_j dx^j$, the *electromagnetic potential*. The electric and magnetic fields are then components of the electromagnetic *field tensor* F defined by

$$F = dA \qquad \text{or, in components,} \qquad F_{jk} \;=\; \partial_j A_k - \partial_k A_j\,.$$

The Maxwell equations take the form

$$\partial_k F^{kl} \;=\; C\, j^l\,, \tag{1.1.2}$$

where j is the so-called 4-current and the constant $C = 4\pi e$ involves the electromagnetic coupling constant (we use the sign convention where $e > 0$). For an observer in a reference frame, the time and spatial components j^0 and $\vec{j}$ of the 4-current have the interpretation as the electric charge density and the electric current density, respectively. Since we shall always work in the 4-dimensional setting, it is unambiguous to refer to j simply as the *electromagnetic current*. Now consider a point particle of mass m and unit charge e in a given (=external) electromagnetic field. Since by causality its velocity is always smaller than light speed, its trajectory is a timelike curve $q(\tau)$. Thus we can parametrize it by the arc length, i.e.

$$\langle u, u\rangle \equiv 1 \qquad \text{with} \qquad u(\tau) := \frac{d}{d\tau} q(\tau)\,.$$

In a reference frame, the time and spatial components of the vector $m \cdot u(\tau)$ are the energy and momentum of the particle at the space-time point $q(\tau)$. We refer to mu as the *momentum* of the particle. The parameter τ has the interpretation as the *proper time* of an observer moving along q. The equation of motion is the tensor equation $m\frac{d}{d\tau}u^j = -eF^{jk}u_k$. Since we shall only consider particles of unit charge, it is convenient to remove the parameter e from the equation of motion. To this end, we rescale the electromagnetic potential according to $A \to e^{-1}A$. Then the equation of

motion simplifies to

$$m \frac{d}{d\tau} u^j \;=\; -F^{jk}\, u_k \,, \tag{1.1.3}$$

whereas the constant C in the Maxwell equations (1.1.2) becomes $C = 4\pi e^2$.

The starting point for general relativity is the observation that a physical process involving gravity can be understood in different ways. Consider for example an observer at rest on earth looking at a freely falling person (e.g. a diver who just jumped from a diving board). The observer at rest argues that the earth's gravitational force, which he can feel himself, also acts on the freely falling person and accelerates him. The person at free fall, on the other hand, does not feel gravity. He can take the point of view that he himself is at rest, whereas the earth is accelerated towards him. He then concludes that there are no gravitational fields, and that the observer on earth merely feels the force of inertia corresponding to his acceleration. Einstein postulated that these two points of view should be equivalent descriptions of the physical process. More generally, it depends on the observer whether one has a gravitational force or an inertial force. In other words,

equivalence principle: no physical experiment can distinguish between gravitational and inertial forces.

In mathematical language, observers correspond to coordinate systems, and so the equivalence principle states that the physical equations should be formulated in general (i.e. "curvilinear") coordinate systems, and should in all these coordinate systems have the same mathematical structure. This means that the physical equations should be invariant under diffeomorphisms, and thus space-time is to be modeled by a *Lorentzian manifold* (M, g).

A Lorentzian manifold is "locally Minkowski space" in the sense that at every space-time point $p \in M$, the corresponding *tangent space* T_pM is a vector space endowed with a scalar product $\langle .,. \rangle_p$ of signature $(+ - - -)$. Therefore, we can distinguish between spacelike, timelike and null tangent vectors. Defining a non-spacelike curve $q(\tau)$ by the condition that its tangent vector $u(\tau) \in T_{q(\tau)}M$ be everywhere non-spacelike, our above definition of light cones and the notion of causality immediately carry over to a Lorentzian manifold. In a coordinate chart, the scalar product $\langle .,. \rangle_p$ can be represented in the form (1.1.1) where g_{jk} is the so-called *metric tensor*. In contrast to Minkowski space, the metric tensor is not a constant matrix but depends on the space-time point, $g_{jk} = g_{jk}(p)$. Its ten components can be regarded as the relativistic analogue of Newton's gravitational potential. For every $p \in M$ there are coordinate systems in which the metric tensor coincides with the Minkowski metric up to second order,

$$g_{jk}(p) \;=\; \mathrm{diag}(1, -1, -1, -1) \,, \qquad \partial_j g_{kl}(p) \;=\; 0 \,. \tag{1.1.4}$$

Such *Gaussian normal coordinates* correspond to the reference frame of a "freely falling observer" who feels no gravitational forces. However, it is in general impossible to arrange that also $\partial_{jk} g_{lm}(p) = 0$. This means that by going into a suitable reference frame, the gravitational field can be transformed away locally (=in one point), but not globally. With this in mind, a reference frame corresponding to Gaussian normal coordinates is also called a *local inertial frame*.

The equation of motion (1.1.3) and the Maxwell equations (1.1.2) can easily be formulated on a Lorentzian manifold by the prescription that they should in a local

inertial frame have the same form as in Minkowski space; this is referred to as the *strong equivalence principle*. It amounts to replacing all partial derivatives by the corresponding *covariant derivatives* ∇ of the Levi-Civita connection; we write symbolically

$$\partial \;\longrightarrow\; \nabla\,. \tag{1.1.5}$$

We thus obtain the equations

$$m\,\nabla_\tau u^j \;=\; -F^{jk} u_k\,, \qquad\qquad \nabla_k F^{kl} \;=\; 4\pi e^2\, j^l \tag{1.1.6}$$

with $F_{jk} = (dA)_{jk} = \nabla_j A_k - \nabla_k A_j$.

The gravitational field is described via the curvature of space-time. More precisely, the Riemannian *curvature tensor* is defined by the relations

$$R^i_{jkl}\, u^l \;=\; \nabla_j \nabla_k u^i - \nabla_k \nabla_j u^i\,. \tag{1.1.7}$$

Contracting indices, one obtains the *Ricci tensor* $R_{jk} = R^i_{jik}$ and *scalar curvature* $R = R^j_j$. The relativistic generalization of Newton's gravitational law are the Einstein equations

$$R_{jk} - \frac{1}{2}\, R\, g_{jk} \;=\; 8\pi\kappa\, T_{jk}\,, \tag{1.1.8}$$

where κ is the gravitational constant. Here the *energy-momentum tensor* T_{jk} gives the distribution of matter and energy in space-time.

It is very convenient that the physical equations can all be derived from a variational principle. To this end, one considers the *action* (see e.g. [**LL**])

$$S \;=\; \int \Big(m\, g_{jk}\, u^j u^k + A_j u^j\Big)\, d\tau \;+\; \int_M \left(-\frac{1}{16\pi e^2}\, F_{jk} F^{jk} - \frac{1}{16\pi\kappa}\, R\right) d\mu\,, \tag{1.1.9}$$

where $u = c'(\tau)$ is the tangent vector of a timelike curve, and $d\mu := \sqrt{-\det g}\; d^4x$ is the integration measure on M. This action is not bounded below, but one can nevertheless look for stationary points and derive the corresponding Euler-Lagrange equations. Varying the space-time curve, the electromagnetic potential and the metric yield the equations of motion, the Maxwell equations and the Einstein equations, respectively.

1.2. Relativistic Quantum Mechanics

We now give an elementary introduction to relativistic quantum mechanics in Minkowski space (for more details see [**BD1, T**]). According to the Heisenberg Uncertainty Principle, the position and momentum of a quantum mechanical particle cannot be determined simultaneously, making it impossible to describe the particle by a trajectory in space-time. Instead, one uses a *wave function* $\Psi(t, \vec{x})$, whose absolute square has the interpretation as the probability density that the particle is at position $\vec{x}$. The simplest relativistic wave equation is the *Klein-Gordon equation*

$$(-\Box - m^2)\, \Psi \;=\; 0\,, \tag{1.2.1}$$

where $\Box \equiv \partial_j \partial^j$ is the wave operator. This equation describes a scalar particle (=particle without spin) of mass m. If the particle has electric charge, one needs to suitably insert the electromagnetic potential A into the Klein-Gordon equation. More precisely, one finds empirically that the equation

$$-(\partial_k - iA_k)(\partial^k - iA^k)\, \Psi \;=\; m^2 \Psi \tag{1.2.2}$$

describes a scalar particle of mass m and charge e in the presence of an electromagnetic field.

In order to describe a particle with spin, it was Dirac's idea to work with a first order differential operator whose square is the wave operator. One introduces the *Dirac matrices* γ^j as 4×4-matrices which satisfy the *anti-commutation relations*

$$2\, g^{jk}\, \mathbb{1} \;=\; \{\gamma^j,\, \gamma^k\} \;\equiv\; \gamma^j \gamma^k + \gamma^k \gamma^j \,. \tag{1.2.3}$$

Then the square of the operator $\gamma^j \partial_j$ is

$$(\gamma^j \partial_j)^2 \;=\; \gamma^j \gamma^k\, \partial_j \partial_k \;=\; \frac{1}{2}\, \{\gamma^j, \gamma^k\}\, \partial_{jk} \;=\; \Box \,. \tag{1.2.4}$$

For convenience, we shall always work in the Dirac representation

$$\gamma^0 \;=\; \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \qquad \vec{\gamma} \;=\; \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}, \tag{1.2.5}$$

where $\vec{\sigma}$ are the three Pauli matrices

$$\sigma^1 \;=\; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 \;=\; \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 \;=\; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The Dirac equation in the vacuum reads

$$\left(i\gamma^k \frac{\partial}{\partial x^k} - m \right) \Psi(x) \;=\; 0 \,, \tag{1.2.6}$$

where $\Psi(x)$, the *Dirac spinor*, has four complex components. The leptons and quarks in the standard model are Dirac particles, and thus one can say that all matter is on the fundamental level described by the Dirac equation. Multiplying (1.2.6) by the operator $(i\gamma^j \partial_j + m)$ and using (1.2.4), one finds that each component of Ψ satisfies the Klein-Gordon equation (1.2.1). In the presence of an electromagnetic field, the Dirac equation must be modified to

$$i\gamma^k (\partial_k - iA_k) \Psi \;=\; m\Psi \,. \tag{1.2.7}$$

Multiplying by the operator $(i\gamma^j(\partial_j - iA_j) + m)$ and using the anti-commutation relations, we obtain the equation

$$\left[-(\partial_k - iA_k)(\partial^k - iA^k) + \frac{i}{2}\, F_{jk} \gamma^j \gamma^k - m^2 \right] \Psi \;=\; 0 \,.$$

This differs from the Klein-Gordon equation (1.2.2) by the extra term $\frac{i}{2} F_{jk} \gamma^j \gamma^k$, which describes the coupling of the spin to the electromagnetic field. We also denote the contraction with Dirac matrices by a slash, i.e. $\not{u} = \gamma^j u_j$ for u a vector of Minkowski space and $\not{\partial} = \gamma^j \partial_j$.

The wave functions at every space-time point are endowed with an indefinite scalar product of signature $(2,2)$, which we call *spin scalar product* and denote by

$$\prec \Psi \,|\, \Phi \succ (x) \;=\; \sum_{\alpha=1}^{4} s_\alpha\, \Psi^\alpha(x)^*\, \Phi^\alpha(x) \,, \qquad s_1 = s_2 = 1,\; s_3 = s_4 = -1 \,, \tag{1.2.8}$$

where Ψ^* is the complex conjugate wave function (this scalar product is often written as $\overline{\Psi}\Phi$ with the so-called adjoint spinor $\overline{\Psi} \equiv \Psi^* \gamma^0$). By the *adjoint* A^* of a matrix A we always mean the adjoint with respect to the spin scalar product as defined via the relations

$$\prec A^* \Psi \,|\, \Phi \succ \;=\; \prec \Psi \,|\, A\Phi \succ \qquad \text{for all } \Psi, \Phi.$$

In an obvious way, this definition of the adjoint gives rise to the notions "*selfadjoint*," "anti-selfadjoint" and "*unitary*." With these notions, the Dirac matrices are selfadjoint, meaning that

$$\prec \gamma^l \Psi \,|\, \Phi \succ \;=\; \prec \Psi \,|\, \gamma^l \Phi \succ \qquad \text{for all } \Psi, \Phi.$$

To every solution Ψ of the Dirac equation we can associate a time-like vector field J by

$$J^k \;=\; \prec \Psi \,|\, \gamma^k\, \Psi \succ\,, \tag{1.2.9}$$

which is called the *Dirac current*. The Dirac current is divergence-free,

$$\begin{aligned} \partial_k J^k &= \partial_k \prec \Psi \,|\, \gamma^k\, \Psi \succ \;=\; \prec \partial_k \Psi \,|\, \gamma^k\, \Psi \succ + \prec \Psi \,|\, \gamma^k \partial_k\, \Psi \succ \\ &= i\,(\prec i \partial\!\!\!/ \Psi \,|\, \Psi \succ - \prec \Psi \,|\, i \partial\!\!\!/ \Psi \succ) \\ &= i\,(\prec (i \partial\!\!\!/ + A\!\!\!/ - m) \Psi \,|\, \Psi \succ - \prec \Psi \,|\, (i \partial\!\!\!/ + A\!\!\!/ - m) \Psi \succ) \;=\; 0\,, \end{aligned}$$

this is referred to as *current conservation*.

So far Dirac spinors were introduced in a given reference frame. In order to verify that our definitions are coordinate independent, we consider two reference frames (x^j) and $(\tilde{x}^l)$ with the same orientation of time and space. Then the reference frames are related to each other by an orthochronous proper Lorentz transformation Λ, i.e. in components

$$\tilde{x}^l \;=\; \Lambda^l_j\, x^j\,, \qquad\qquad \Lambda^l_j\, \frac{\partial}{\partial \tilde{x}^l} \;=\; \frac{\partial}{\partial x^j}\,,$$

and Λ leaves the Minkowski metric invariant,

$$\Lambda^l_j\, \Lambda^m_k\, g_{lm} \;=\; g_{jk}\,. \tag{1.2.10}$$

Under this change of space-time coordinates, the Dirac operator $i\gamma^j(\partial_{x^j} - iA_j)$ transforms to

$$i\tilde{\gamma}^l \left(\frac{\partial}{\partial \tilde{x}^l} - i\tilde{A}_l \right) \qquad \text{with} \qquad \tilde{\gamma}^l \;=\; \Lambda^l_j \gamma^j\,. \tag{1.2.11}$$

This transformed Dirac operator does not coincide with the Dirac operator $i\gamma^l(\partial_{\tilde{x}^l} - i\tilde{A}_l)$ as defined in the reference frame $(\tilde{x}^l)$ because the Dirac matrices have a different form. However, the next lemma shows that the two Dirac operators do coincide up to a suitable unitary transformation of the spinors.

LEMMA 1.2.1. *For any orthochronous proper Lorentz transformation Λ there is a unitary matrix $U(\Lambda)$ such that*

$$U(\Lambda)\, \Lambda^l_j \gamma^j\, U(\Lambda)^{-1} \;=\; \gamma^l\,.$$

Proof. Since Λ is orthochronous and proper, we can write it in the form $\Lambda = \exp(\lambda)$, where λ is a suitable generator of a rotation and/or a Lorentz boost. Then $\Lambda(t) := \exp(t\lambda)$, $t \in \mathbb{R}$, is a family of Lorentz transformations, and differentiating (1.2.10) with respect to t at $t = 0$, we find that

$$\lambda^l_j\, g_{lk} \;=\; -g_{jm}\, \lambda^m_k\,.$$

Using this identity together with the fact that the Dirac matrices are selfadjoint, it is straightforward to verify that the matrix

$$u \;:=\; \frac{1}{4}\, \lambda^l_k\, \gamma_l \gamma^k$$

is anti-selfadjoint. As a consequence, the family of matrices

$$U(t) \;:=\; \exp(tu)$$

is unitary. We now consider for a fixed index l the family of matrices

$$A(t) \;:=\; U(t)\, \Lambda(t)^l_j \gamma^j\, U(t)^{-1}\,.$$

Clearly, $A(0) = \gamma^l$. Furthermore, differentiating with respect to t gives

$$\frac{d}{dt}\, A(t) \;=\; U\, \Lambda^l_j \left\{ u\, \gamma^j - \gamma^j\, u + \lambda^j_k \gamma^k \right\} U^{-1}\,,$$

and a short calculation using the commutation relations

$$\left[\gamma_l \gamma_k, \gamma^j\right] \;=\; 2\left(\gamma_l\, g^{kj} - \delta^j_l\, \gamma^k\right)$$

shows that the curly brackets vanish. We conclude that $A(1) = A(0)$, proving the lemma. ∎

Applying this lemma to the Dirac operator in (1.2.11), one sees that the Dirac operator is invariant under the joint transformation of the space-time coordinates and the spinors

$$x^j \;\longrightarrow\; \Lambda^j_k x^k\,, \qquad \Psi \;\longrightarrow\; U(\Lambda)\, \Psi\,. \tag{1.2.12}$$

Moreover, since the matrix $U(\Lambda)$ is unitary, the representation of the spin scalar product (1.2.8) is valid in any reference frame. We conclude that our definition of spinors is indeed coordinate invariant.

Out of the Dirac matrices one can form the *pseudoscalar matrix* ρ by

$$\rho \;=\; \frac{i}{4!}\, \epsilon_{jklm} \gamma^j \gamma^k \gamma^l \gamma^m \tag{1.2.13}$$

(this matrix in the physics literature is usually denoted by γ^5). Here ϵ_{jklm} is the totally antisymmetric symbol (i.e. ϵ_{jklm} is equal to ± 1 if (j,k,l,m) is an even and odd permutation of $(0,1,2,3)$, respectively, and vanishes otherwise). A short calculation shows that the pseudoscalar matrix is anti-selfadjoint and $\rho^2 = \mathbb{1}$. As a consequence, the matrices

$$\chi_L \;=\; \frac{1}{2}\,(\mathbb{1} - \rho)\;, \qquad \chi_R \;=\; \frac{1}{2}\,(\mathbb{1} + \rho) \tag{1.2.14}$$

satisfy the relations

$$\chi_{L/R}^2 = \chi_{L/R}\,, \qquad \rho\, \chi_L = -\chi_L\,, \qquad \rho\, \chi_R = \chi_R\,, \qquad \chi_L^* \;=\; \chi_R\,, \qquad \chi_L + \chi_R = \mathbb{1}\,.$$

They can be regarded as the spectral projectors of the matrix ρ and are called the *chiral projectors* . The projections $\chi_L \Psi$ and $\chi_R \Psi$ are referred to as the *left-* and *right-handed* components of the spinor. A matrix is said to be *even* and *odd* if it commutes and anti-commutes with ρ, respectively. It is straightforward to verify that the Dirac matrices are odd, and therefore

$$\gamma^j\, \chi_{L/R} \;=\; \chi_{R/L}\, \gamma^j\,.$$

Using this relation, one can rewrite the Dirac equation (1.2.7) as a system of equations for the left- and right-handed components of Ψ,

$$i\gamma^k (\partial_k - iA_k)\, \chi_L \Psi \;=\; m\, \chi_R \Psi\,, \qquad i\gamma^k (\partial_k - iA_k)\, \chi_R \Psi \;=\; m\, \chi_L \Psi\,.$$

If $m = 0$, these two equations decouple, and we get separate equations for the left- and right-handed components of Ψ. This observation is the starting point of the 2-component Weyl spinor formalism. We shall not use this formalism here, but will instead describe chiral massless particles (like neutrinos) by the left- or right-handed component of a Dirac spinor.

For the probabilistic interpretation of the Dirac wave function, we need to distinguish a direction of time and work in a particular reference frame. Then the zero component of the Dirac current $J^0(t, \vec{x})$ has for a given time t the interpretation as the *probability density* of the particle to be at position $\vec{x}$ (and is thus the relativistic analogue of the absolute square $|\Psi|^2$ of the Schrödinger or Pauli wave functions). Clearly, for this probabilistic interpretation the wave function must be properly normalized. More precisely, physical states must satisfy the *normalization condition*

$$\int_{\mathbb{R}^3} \prec \Psi \,|\, \gamma^0\, \Psi \succ (t, \vec{x})\, d\vec{x} \;=\; 1\,. \tag{1.2.15}$$

The integral in (1.2.15) is also called the *probability integral*. Using Gauss' (divergence) theorem and the current conservation, one sees that the normalization integral is time independent,

$$\begin{aligned} &\int_{\mathbb{R}^3} \prec \Psi \,|\, \gamma^0\, \Psi \succ (t_2, \vec{x})\, d\vec{x} - \int_{\mathbb{R}^3} \prec \Psi \,|\, \gamma^0\, \Psi \succ (t_1, \vec{x})\, d\vec{x} \\ &= \int_{t_1}^{t_2} dt \int_{\mathbb{R}^3} d\vec{x}\; \partial_k \prec \Psi \,|\, \gamma^k\, \Psi \succ (t, \vec{x}) \;=\; 0\,, \end{aligned} \tag{1.2.16}$$

and thus it suffices to satisfy (1.2.15) for example at $t = 0$.

In a given reference frame, it is convenient to introduce a positive scalar product by polarizing the normalization integral,

$$(\Psi \,|\, \Phi) \;:=\; \int_{\mathbb{R}^3} \prec \Psi \,|\, \gamma^0\, \Phi \succ (t, \vec{x})\, d\vec{x}\,. \tag{1.2.17}$$

We denote the Hilbert space corresponding to this scalar product by $\mathcal{H} = L^2(\mathbb{R}^3)^4$. Multiplying the Dirac equation (1.2.7) by γ^0 and bringing the t-derivative on a separate side of the equation, we can write the Dirac equation as

$$i\partial_t \Psi \;=\; h\Psi \tag{1.2.18}$$

with a purely spatial operator h. Clearly, this equation is not manifestly covariant. In analogy to nonrelativistic quantum mechanics, it is referred to as the Dirac equation in *Hamiltonian form*, and h is the *Hamiltonian*. If Ψ and Φ are solutions of the Dirac equation, one sees similar to (1.2.16) that the scalar product (1.2.17) is independent of time. Hence

$$0 \;=\; \partial_t (\Psi \,|\, \Phi) \;=\; i\,((h\Psi \,|\, \Phi) - (\Psi \,|\, h\Phi))\;.$$

This shows that the Hamiltonian is a symmetric operator on $\mathcal{H}$.

We conclude this section by a brief discussion of the solutions of the free Dirac equation (=the Dirac equation without electromagnetic field) in the case $m \neq 0$. Taking the Fourier transform of the wave function,

$$\Psi(x) \;=\; \int \frac{d^4k}{(2\pi)^4}\, \hat{\Psi}(k)\, e^{-ikx}\;, \qquad \hat{\Psi}(k) \;=\; \int d^4x\, \Psi(x)\, e^{ikx}\;,$$

the Dirac equation $(i\partial\!\!\!/ - m)\Psi = 0$ reduces to the algebraic equation in momentum space

$$(k\!\!\!/ - m)\, \hat{\Psi}(k) \;=\; 0\,. \tag{1.2.19}$$

Multiplying by $k\!\!\!/ + m$ and using the identity $(k\!\!\!/ - m)(k\!\!\!/ + m) = k^2 - m^2$, one sees that if $k^2 \neq m^2$, the matrix $k\!\!\!/ - m$ is invertible and thus (1.2.19) has no solutions. If conversely $k^2 = m^2$, we have the relation $(k\!\!\!/ - m)^2 = -2m(k\!\!\!/ - m)$, showing that the matrix $k\!\!\!/ - m$ is diagonalizable and that its eigenvalues are either $-2m$ or zero. Taking the trace, $\mathrm{Tr}(k\!\!\!/ - m) = -4m$, it follows that the matrix $k\!\!\!/ - m$ has a two-dimensional kernel. A short calculation shows that the projector onto this kernel is given by

$$\Pi(k) \;:=\; \frac{k\!\!\!/ + m}{2m}\,. \tag{1.2.20}$$

We conclude that (1.2.19) has a solution only if k is on the *mass shell* $\{k \mid k^2 = m^2\}$. For each k on the mass shell, (1.2.19) has exactly two linearly independent solutions. In order to give these solutions more explicitly, we choose a reference frame $(t, \vec{x})$ and denote the corresponding momentum variables by $k = (\omega, \vec{p})$. The momenta on the mass shell are then given by

$$\omega \;=\; \omega(\vec{p}, \epsilon) := \epsilon \sqrt{|\vec{p}|^2 + m^2}$$

with parameters $\vec{p} \in \mathbb{R}^3$ and $\epsilon \in \{\pm 1\}$. The momenta with $\epsilon = 1$ and $\epsilon = -1$ are said to be on the *upper* and *lower mass shell*, respectively. For any given $(\vec{p}, \epsilon)$, we label the two linearly independent solutions of (1.2.19) by a parameter $s \in \{1, 2\}$ and denote them by $\chi_{\vec{p}s\epsilon}$. It is most convenient to choose them pseudo-orthonormal with respect to the spin scalar product,

$$\prec \chi_{\vec{p}s\epsilon}, \chi_{\vec{p}s'\epsilon} \succ \;=\; \epsilon\, \delta_{s,s'} \qquad \text{for all } \vec{p} \in \mathbb{R}^3, \epsilon \in \{\pm 1\} \text{ and } s, s' \in \{1, 2\}\,. \tag{1.2.21}$$

Here the factor ϵ reflects that the solution spaces on the upper and lower mass shell are positive and negative definite, respectively. Using a bra/ket notation in the spin scalar product, we get a simple representation of the projector (1.2.20),

$$\sum_{s=1,2} \epsilon\, |\chi_{\vec{p}s\epsilon} \succ \prec \chi_{\vec{p}s\epsilon}| \;=\; \Pi(\omega(\vec{p}, \epsilon), \vec{p})\,. \tag{1.2.22}$$

The spinors $\chi_{\vec{p}s\epsilon}$ form a complete set of solutions of (1.2.19). Taking their suitably normalized Fourier transform, we obtain the *plane wave solutions*

$$\Psi_{\vec{p}s\epsilon}(t, \vec{x}) \;=\; \frac{1}{(2\pi)^{\frac{3}{2}}}\, e^{-i\omega(\vec{p},\epsilon)t + i\vec{p}\vec{x}}\, \chi_{\vec{p}s\epsilon}\,. \tag{1.2.23}$$

Each solution of the free Dirac equation is a superposition of plane wave solutions.

In the Hamiltonian framework (1.2.18), the plane wave solutions are eigenfunctions of the Hamiltonian with eigenvalue $\omega(\vec{p}, \epsilon)$,

$$h\, \Psi_{\vec{p}s\epsilon} \;=\; \omega(\vec{p}, \epsilon)\, \Psi_{\vec{p}s\epsilon}\,.$$

Since the eigenvalue of the Hamiltonian has the interpretation as the physical energy of the state, we conclude that the plane wave solutions on the upper and lower mass shell have positive and negative energy, respectively. Expressed more mathematically, the plane wave solutions correspond to points in the essential spectrum of the Hamiltonian, and thus

$$\sigma_{\text{ess}}(h) \;=\; \{\pm\sqrt{\vec{p}^2 + m^2},\, \vec{p} \in \mathbb{R}^3\} \;=\; (-\infty, -m] \cup [m, \infty)\,.$$

In particular, we conclude that the Dirac equation has solutions of negative energy and that the Hamiltonian is not bounded below. This was originally considered a serious problem of Dirac theory, mainly because a system with unbounded Hamiltonian has no stable ground state. Dirac resolved these problems by introducing the so-called Dirac sea [**D2**]. The concept of the Dirac sea plays a crucial role in the present work. At this point, we merely explain Dirac's idea in words (in the next section we shall explain how it is implemented mathematically in the framework of quantum field theory, and in Chapter §2 we will come back to it in greater detail). Thinking of many-particle quantum mechanics (and assuming that the particles do not interact with each other), the solutions of the Dirac equation can be regarded as one-particle states, which can be occupied by the particles of the system. According to the Pauli Exclusion Principle, each state may be occupied by at most one particle. If one assumes that no states are occupied in the vacuum, a system of n particles is unstable because the energy of the system can be made negative and arbitrarily small by occupying n negative-energy states. However, this problem disappears if one assumes that in the vacuum all states of negative energy are already occupied. Then the n additional particles must occupy states of positive energy, and the system becomes stable. This consideration led Dirac to no longer think of the vacuum as "empty space," but instead to conjecture that the vacuum should be formed of a "sea" of quantum mechanical particles of negative energy. Dirac's conception was that the effects of all the particles of the sea should compensate each other in such a way that the sea cannot be observed. Likewise, in this picture an interacting system of n particles corresponds to the Dirac sea and n additional particles of positive energy which all interact with each other. This intuitive concept of the Dirac sea as a "sea of interacting particles" was not only useful for resolving the problem of the negative-energy solutions, but furthermore led to the prediction of anti-particles and pair creation/annihilation. To this end, Dirac considered an interacting system which at initial time $t = 0$ is the vacuum. Then at a later time, one of the particles of the sea may no longer occupy a state of negative energy, but be instead in a positive-energy state. In this case, the system consists of one particle and one "hole" in the Dirac sea. Since the completely filled Dirac sea should be invisible, the hole appears as a virtual particle of energy and electric charge opposite to that of the unoccupied negative-energy state. Thus the virtual particle has positive energy, but its charge is opposite to that of an ordinary particle. This virtual particle is referred to as anti-particle. In the above process, particles and anti-particles are always generated in pairs, this explains the physical effect of pair creation. Conversely, a particle and a hole can recombine in a process called pair annihilation.

1.3. Fock Space Quantization of the Free Dirac Field

In this section we outline the canonical quantization of the free Dirac field (for details see [**BD2, IZ**]). For clarity, we first quantize without taking into account the Dirac sea and explain afterwards how the construction is to be modified in order to cure the problem of the negative-energy states. We begin with the one-particle Hilbert space $(\mathcal{H}, (.,.))$ in the Hamiltonian framework (1.2.18). Clearly, the plane-wave solutions (1.2.23) are not square integrable, but we can normalize them in the distributional sense. More precisely,

$$(\Psi_{\vec{p}s\epsilon} \,|\, \Psi_{\vec{p}'s'\epsilon'}) \;=\; \delta^3(\vec{p}-\vec{p'})\, \prec\chi_{\vec{p}s\epsilon} \,|\, \gamma^0 \chi_{\vec{p}s'\epsilon'}\succ . \tag{1.3.1}$$

In order to compute the inner product $\prec \chi_{\vec{p}s\epsilon} \,|\, \gamma^0 \chi_{\vec{p}s'\epsilon'} \succ$, we first plug in the spectral projectors (1.2.20), which for convenience we now denote by $\Pi_{\vec{p}\epsilon} := \Pi(\omega(\vec{p},\epsilon),\vec{p})$,

$$\prec \chi_{\vec{p}s\epsilon} \,|\, \gamma^0 \chi_{\vec{p}s'\epsilon'} \succ \;=\; \prec \Pi_{\vec{p}\epsilon} \chi_{\vec{p}s\epsilon} \,|\, \gamma^0\, \Pi_{\vec{p}\epsilon'} \chi_{\vec{p}s'\epsilon'} \succ \;=\; \prec \chi_{\vec{p}s\epsilon} \,|\, \Pi_{\vec{p}\epsilon}\, \gamma^0\, \Pi_{\vec{p}\epsilon'} \chi_{\vec{p}s'\epsilon'} \succ .$$

The matrix product $\Pi_{\vec{p}\epsilon} \gamma^0 \Pi_{\vec{p}\epsilon'}$ is computed in the cases $\epsilon = \epsilon'$ and $\epsilon \neq \epsilon'$ as follows,

$$\begin{aligned}
\Pi_{\vec{p}\epsilon}\, \gamma^0\, \Pi_{\vec{p}\,-\epsilon} &= \frac{\omega\gamma^0 - \vec{p}\vec{\gamma} + m}{2m}\, \gamma^0\, \frac{-\omega\gamma^0 - \vec{p}\vec{\gamma} + m}{2m} \\
&= \frac{(\not{k} + m)(-\not{k} + m)}{4m^2}\, \gamma^0 \;=\; 0 \\
\Pi_{\vec{p}\epsilon}\, \gamma^0\, \Pi_{\vec{p}\epsilon} &= \frac{\omega\gamma^0 - \vec{p}\vec{\gamma} + m}{2m}\, \gamma^0\, \frac{\omega\gamma^0 - \vec{p}\vec{\gamma} + m}{2m} \\
&= \frac{\omega\gamma^0 - \vec{p}\vec{\gamma} + m}{2m}\, 2\omega \;+ \frac{(\not{k} + m)(-\not{k} + m)}{4m^2}\, \gamma^0 \;=\; 2\omega\, \Pi_{\vec{p}\epsilon}\,,
\end{aligned}$$

where we set $\omega = |\omega(\vec{p},\epsilon)|$ and $k = (\omega, \vec{p})$. Hence the matrix products reduce to a multiple of the identity, and we can use the normalization (1.2.21) to obtain

$$(\Psi_{\vec{p}s\epsilon} \,|\, \Psi_{\vec{p}'s'\epsilon'}) \;=\; 2\omega(\vec{p})\, \delta^3(\vec{p} - \vec{p}')\, \delta_{\epsilon\epsilon'}\, \delta_{ss'}\,, \tag{1.3.2}$$

with $\omega(\vec{p}) := |\omega(\vec{p},\epsilon)| = \sqrt{|\vec{p}|^2 + m^2}$. Readers who dislike this δ-normalization can also state (1.3.2) by saying that, similar to a Fourier transformation, the mapping

$$L^2\!\left(\mathbb{R}^3, \frac{d\vec{p}}{2\omega(\vec{p})}\right)^4 \;\longrightarrow\; \mathcal{H} \;:\; f_{s\epsilon}(\vec{p}) \;\longmapsto\; \sum_{s,\epsilon} \int_{\mathbb{R}^3} \frac{d\vec{p}}{2\omega(\vec{p})}\, f_{s\epsilon}(\vec{p})\, \Psi_{\vec{p}s\epsilon}(t,\vec{x})$$

is an isometry of Hilbert spaces. The factor $d\vec{p}/(2\omega(\vec{p}))$ appearing here can be interpreted as the Lorentz invariant measure on the mass shell (i.e. formally $d\vec{p}/(2\omega(\vec{p})) = \delta(k^2 - m^2)\, d^4k$), and we abbreviate it in what follows by $d\mu_{\vec{p}}$.

In many-particle quantum mechanics, the system where the n one-particle states $\Psi_{\vec{p}_1 s_1 \epsilon_1}, \ldots, \Psi_{\vec{p}_n s_n \epsilon_n}$ are occupied is described by the *Hartree-Fock state*

$$\Psi \;=\; \Psi_{\vec{p}_1 s_1 \epsilon_1} \,\wedge\, \cdots \,\wedge\, \Psi_{\vec{p}_n s_n \epsilon_n}\,. \tag{1.3.3}$$

Here the wedge product $\wedge$ is the anti-symmetrized tensor product. Due to the anti-symmetry, the wedge product vanishes if two of the one-particle wave functions $\Psi_{\vec{p}_i, s_i, \epsilon_i}$ coincide. This corresponds to the

Pauli Exclusion Principle:	each quantum mechanical state can be occupied by at most one particle.

Particles which obey the Pauli Exclusion Principle are called *fermions* (whereas for bosons one uses instead of (1.3.3) the symmetric tensor product). Working with the n-particle state (1.3.3) also implies that the n particles are *indistinguishable* in the sense that if we exchange two particles, the wave function Ψ changes only by a physically irrelevant phase.

A general n-particle state corresponds to a linear combination of Hartree-Fock states and is thus a vector of the Hilbert space $\mathcal{F}^n = \wedge^n \mathcal{H}$. In quantum field theory, the number of particles is not fixed, and therefore the Dirac particles are described more generally by a vector of the *fermionic Fock space* $\mathcal{F} = \oplus_{n=0}^\infty \mathcal{F}^n$. Notice that the

scalar product on $\mathcal{F}$ is induced by that on $\mathcal{H}$; we denote it for clarity by $(.|.)_{\mathcal{F}}$. On the Fock space, we introduce the field operators $\hat{\Psi}^\dagger_{\vec{p}s\epsilon}$ by

$$\hat{\Psi}^\dagger_{\vec{p}s\epsilon} \;:\; \mathcal{F}^n \;\longrightarrow\; \mathcal{F}^{n+1} \;:\; \Psi \;\longmapsto\; \Psi_{\vec{p}s\epsilon} \wedge \Psi$$

and denote their adjoint with respect to the scalar product $(.|.)_{\mathcal{F}}$ by $\hat{\Psi}_{\vec{p}s\epsilon}$, $\hat{\Psi}_{\vec{p}s\epsilon} = (\hat{\Psi}^\dagger_{\vec{p}s\epsilon})^*$. The operators $\hat{\Psi}^\dagger_{\vec{p}s\epsilon}$ and $\hat{\Psi}_{\vec{p}s\epsilon}$ are referred to as the *creation* and *annihilation operators*, respectively. A straightforward calculation using our definitions and the normalization condition (1.3.2) yields that the field operators satisfy the *canonical anticommutation relations*

$$\begin{aligned} \{\hat{\Psi}_{\vec{p}s\epsilon}, \hat{\Psi}_{\vec{p}'s'\epsilon'}\} &= 0 = \{\hat{\Psi}^\dagger_{\vec{p}s\epsilon}, \hat{\Psi}^\dagger_{\vec{p}'s'\epsilon'}\} \\ \{\hat{\Psi}_{\vec{p}s\epsilon}, \hat{\Psi}^\dagger_{\vec{p}'s'\epsilon'}\} &= 2\omega(\vec{p})\, \delta^3(\vec{p}-\vec{p}')\, \delta_{\epsilon\epsilon'}\, \delta_{ss'}\;. \end{aligned} \tag{1.3.4}$$

The vacuum corresponding to these field operators, denoted by $|0>$, is a unit vector of $\mathcal{F}$ on which all annihilation operators vanish,

$$\hat{\Psi}_{\vec{p}s\epsilon}\,|0> \;=\; 0 \qquad \text{for all } \vec{p}, s, \epsilon. \tag{1.3.5}$$

The Hartree-Fock states can be obtained from it by applying the creation operators,

$$(1.3.3) \;=\; \hat{\Psi}^\dagger_{\vec{p}_1 s_1 \epsilon_1} \cdots \hat{\Psi}^\dagger_{\vec{p}_n s_n \epsilon_n}\,|0>\;,$$

and taking linear combinations, we can build up the whole Fock space from the vacuum. The energy of the Hartree-Fock state (1.3.3) is the sum of the energies $\omega(\vec{p},\epsilon)$ of all particles, and a short calculation shows that this coincides with the eigenvalue of the following operator on the Fock space,

$$H_0 \;=\; \sum_{\epsilon,s} \int_{\mathbb{R}^3} \omega(\vec{p},\epsilon)\; \hat{\Psi}^\dagger_{\vec{p}s\epsilon} \hat{\Psi}_{\vec{p}s\epsilon}\, d\mu_{\vec{p}}\,. \tag{1.3.6}$$

Thus H_0 is the Hamiltonian of the free many-particle theory.

Since the factor $\omega(\vec{p},\epsilon)$ in the integrand can be negative, the Hamiltonian (1.3.6) is not bounded from below. This is precisely the problem of the negative-energy solutions of the Dirac equation which we described at the end of the previous section. This problem disappears in quantum field theory as follows. According to the concept of the Dirac sea, all negative-energy states should be occupied in the vacuum. This is implemented here by redefining the vacuum; namely we replace (1.3.5) by the conditions

$$\hat{\Psi}_{\vec{p}s+}\,|0> \;=\; 0 \;=\; \hat{\Psi}^\dagger_{\vec{p}s-}\,|0> \qquad \text{for all } \vec{p}, s. \tag{1.3.7}$$

Since the anti-particles correspond to "holes" in the Dirac sea, we reinterpret the creation operators for the negative-energy states as annihilation operators and vice versa, i.e. we perform the formal replacements

$$\hat{\Psi}_{\vec{p}s-} \;\longleftrightarrow\; \hat{\Psi}^\dagger_{\vec{p}s-}\;. \tag{1.3.8}$$

This is convenient because after the reinterpretation, the new vacuum (1.3.7) again satisfies the usual conditions (1.3.5). The Hamiltonian (1.3.6) transforms under the replacements (1.3.8) into

$$H_0 \;=\; \sum_{\epsilon,s} \int_{\mathbb{R}^3} \omega(\vec{p})\; \hat{\Psi}^\dagger_{\vec{p}s\epsilon} \hat{\Psi}_{\vec{p}s\epsilon}\, d\mu_{\vec{p}} \;-\; \sum_{s} \int_{\mathbb{R}^3} \omega(\vec{p})\; \{\hat{\Psi}^\dagger_{\vec{p}s-}, \hat{\Psi}_{\vec{p}s-}\}\, d\mu_{\vec{p}}\,. \tag{1.3.9}$$

The first part of this Hamiltonian is positive. Using the anti-commutation relations (1.3.4), one sees that the second term in (1.3.9) is an infinite negative constant. Using the argument that adding a constant to the total energy of a system is nothing more than introducing a new convention for the zero point of energy measurements, one drops this second term and redefines the Hamiltonian by

$$H_0 \;=\; \sum_{\epsilon,s} \int_{\mathbb{R}^3} |\omega(\vec{p},s)|\; \hat{\Psi}^\dagger_{\vec{p}s\epsilon} \hat{\Psi}_{\vec{p}s\epsilon}\, d\mu_{\vec{p}}\,. \tag{1.3.10}$$

This Hamiltonian is positive and vanishes on the vacuum, giving rise to a satisfying physical theory. However, dropping the second summand in (1.3.9) was a problematic step in the construction. We postpone the discussion of this point to §2.2.

1.4. Classical Gauge Theories

We now briefly introduce the framework of local gauge theories (for a more detailed introduction see for example [**Ga**]). In order to avoid confusion between covariant derivatives ∇ and gauge-covariant derivatives D we restrict attention to Minkowski space. The generalization to curved space-time will be described in connection with the Dirac equation in §1.5. The starting point for gauge theories is the observation that changing the electromagnetic potential by the gradient of a real-valued function Λ,

$$A \;\longrightarrow\; A + \partial\Lambda\,, \tag{1.4.1}$$

leaves the field tensor unchanged,

$$F_{jk} \;\longrightarrow\; F_{jk} + \partial_j\partial_k\Lambda - \partial_k\partial_j\Lambda \;=\; F_{jk}\,.$$

The equations of classical electrodynamics (1.1.2, 1.1.3) do not involve the electromagnetic potential, only its field tensor. Therefore, these equations are obviously invariant under the transformation (1.4.1). In the quantum mechanical wave equations (1.2.2, 1.2.7) the electromagnetic potential does appear, but only in combination with a partial derivative in the operators $\partial_k - iA_k$. These operators transform under (1.4.1) as follows,

$$\partial_k - iA_k \;\longrightarrow\; \partial_k - iA_k - i\partial_k\Lambda \;=\; e^{i\Lambda}\,(\partial_k - iA_k)\,e^{-i\Lambda}\,.$$

Writing the transformation law with the multiplication operators $e^{\pm i\Lambda}$ reveals that the equations of quantum mechanics are invariant under (1.4.1) if at the same time the local phase of the wave functions is transformed according to

$$\Psi \;\longrightarrow\; e^{i\Lambda}\,\Psi\,. \tag{1.4.2}$$

Finally, these local phase transformations leave the Dirac current (1.2.9) unchanged. We conclude that classical field theory and relativistic quantum mechanics are invariant under the transformation (1.4.1, 1.4.2), which is referred to as a *local gauge transformation* of electrodynamics. The invariance of the physical equations under local gauge transformations can be interpreted as a physical symmetry, the *local gauge symmetry*.

Extending the above concept leads to the mathematical framework of gauge theories. We first note that the phase factor $e^{i\Lambda}$ in (1.4.2) can be interpreted as the operation of an element of the Lie group $U(1)$ on Ψ. Likewise, the factors $\partial_j\Lambda = ie^{i\Lambda}\partial_j e^{-i\Lambda}$ can be regarded as elements of the corresponding Lie algebra $u(1)$. Since in (1.4.1) this factor is added to the components A_j of the electromagnetic potential, it is natural to also consider the A_j as $u(1)$-valued functions. In generalization, we let the

gauge group $\mathcal{G}$ be an arbitrary Lie group in a given matrix representation on the wave functions (the wave functions may have more than four components; a typical example is $\mathcal{G} = U(p)$ and $\Psi(x) \in \mathbb{C}^4 \otimes \mathbb{C}^p$). The corresponding Lie algebra in its representation on the wave functions is denoted by $\mathfrak{g}$. We introduce the *gauge potentials* A_j as $\mathfrak{g}$-valued functions on M. For any smooth function $U : M \to \mathcal{G}$, the transformation of the wave functions

$$\Psi(x) \;\longrightarrow\; U(x)\, \Psi(x) \tag{1.4.3}$$

is referred to as a *local gauge transformation*. Clearly, partial derivatives of Ψ do not behave well under gauge transformations because we pick up derivatives of U. This problem disappears if instead of partial derivatives we consider *gauge-covariant derivatives*

$$D_j \;=\; \partial_j - iA_j\,, \tag{1.4.4}$$

provided that the gauge potentials transform according to

$$A_j \;\longrightarrow\; U A_j U^{-1} + iU\,(\partial_j U^{-1})\,. \tag{1.4.5}$$

Namely, a short calculation shows that the gauge-covariant derivative behaves under gauge transformations according to

$$D_j \;\longrightarrow\; U\, D_j\, U^{-1}\,, \tag{1.4.6}$$

and thus the gauge-covariant derivatives of Ψ obey the simple transformation rule

$$D_j \Psi \;\longrightarrow\; U\, D_j \Psi\,.$$

Next we need to introduce the gauge potentials into the physical equations and formulate the equations that describe the dynamics of the gauge fields. We just saw that in order to ensure gauge invariance, one should work with gauge-invariant derivatives instead of partial derivatives. The simplest method for making the physical theory gauge invariant is to replace all partial derivatives by the corresponding gauge-invariant derivatives,

$$\partial \;\longrightarrow\; D\,. \tag{1.4.7}$$

This ad-hoc method is in physics called the *minimal coupling* procedure. For the equations of quantum mechanics it can be motivated if one keeps in mind that with a local gauge transformation of the form $U(x) = \mathbb{1} - iA_j\,(x-p)^j + o(x-p)$ we can always arrange that $A(p) = 0$. In this gauge, the gauge-covariant derivatives coincide at p with the partial derivatives, and thus we can state minimal coupling as follows,

> Around each space-time point p there is a gauge such that the quantum mechanical equations coincide at p with the equations without gauge fields. (1.4.8)

In this formulation, minimal coupling can be understood similar to the strong equivalence principle; we only need to replace "coordinate system" by "gauge" and "gravitational field" by "gauge field." In the example of the free Dirac equation $(i\partial\!\!\!/ - m)\Psi = 0$, minimal coupling yields the equation

$$i\gamma^j(\partial_j - iA_j)\,\Psi \;=\; m\,\Psi\,,$$

which describes a behavior of a Dirac particle in the presence of the gauge field. This equation can also be derived by varying Ψ in the corresponding Dirac action

$$S_{\text{D}} \;=\; \int_M \prec\!\Psi \,|\, \big(i\gamma^j(\partial_j - iA_j) - m\big)\,\Psi\!\succ d\mu.$$

In order to get the equations for the gauge field, we construct out of the gauge-covariant derivative the *field tensor* by

$$F_{jk} \;=\; i\,[D_j, D_k] \;=\; \partial_j A_k - \partial_k A_j - i[A_j, A_k]\,.$$

Since its behavior under gauge transformation is simply

$$F_{jk} \;\longrightarrow\; U\,F_{jk}\,U^{-1}\,,$$

we can generalize the action of the electromagnetic field in (1.1.9) by the *Yang-Mills action*

$$S_{\text{YM}} \;=\; -\frac{1}{16\pi e^2}\int_M \text{Tr}(F_{jk}F^{jk})\,d\mu\,,$$

where "Tr" is a suitably normalized matrix trace. The total action is simply the sum of the Dirac and Yang-Mills actions,

$$S \;=\; S_{\text{D}} + S_{\text{YM}}\,.$$

Varying (Ψ, A) we obtain the coupled Dirac-Yang/Mills equations which describe the classical dynamics.

1.5. Dirac Spinors in Curved Space-Time

Dirac spinors are often formulated on a manifold using frame bundles, either an orthonormal frame [**B, Fr**] or a Newman-Penrose null frame [**PR, Ch**]. We here outline an equivalent formulation of spinors in curved space-time in the framework of a $U(2,2)$ gauge theory (for details see [**F2**]). We restrict attention to the Dirac operator in local coordinates; for global issues like topological obstructions for the existence of spin structures see e.g. [**LM**]. We let M be a 4-dimensional manifold (without Lorentz metric) and define the *spinor bundle* SM as a vector bundle over M with fibre $\mathbb{C}^4$. The fibres are endowed with a scalar product $\prec .|.\succ$ of signature $(2,2)$, which is again referred to as the *spin scalar product*. Sections in the spinor bundle are called *spinors* or wave functions. In local coordinates, a spinor is represented by a 4-component complex function on space-time, usually denoted by $\Psi(x)$. Choosing at every space-time point a pseudo-orthonormal basis $(e_\alpha)_{\alpha=1,\ldots,4}$ in the fibres,

$$\prec e_\alpha | e_\beta \succ \;=\; s_\alpha\,\delta_{\alpha\beta}\,, \qquad s_1 = s_2 = 1,\; s_3 = s_4 = -1 \tag{1.5.1}$$

and representing the spinors in this basis, $\Psi = \Psi^\alpha e_\alpha$, the spin scalar product takes again the form (1.2.8). Clearly, the basis (e_α) is not unique, but at every space-point can be transformed according to

$$e_\alpha \;\longrightarrow\; (U^{-1})^\beta_\alpha\, e_\beta\,,$$

where U is an isometry of the spin scalar product, $U \in U(2,2)$. Under this basis transformation the spinors behave as follows,

$$\Psi^\alpha(x) \;\longrightarrow\; U^\alpha_\beta(x)\,\Psi^\beta(x)\,. \tag{1.5.2}$$

Due to the analogy to (1.4.3) we interpret this transformation of the wave functions as a local gauge transformation with gauge group $\mathcal{G} = U(2,2)$. We refer to a choice of the spinor basis (e_α) as a *gauge*.

Our goal is to formulate classical Dirac theory in such a way that the above $U(2,2)$ gauge transformations correspond to a physical symmetry, the $U(2,2)$ *gauge symmetry* . To this end, we shall introduce the Dirac operator as the basic object on M, from which we will later deduce the Lorentz metric and the gauge potentials. We define a

differential operator $\mathcal{D}$ of first order on the wave functions by the requirement that in a chart and gauge it should be of the form

$$\mathcal{D} \;=\; iG^j(x)\frac{\partial}{\partial x^j} + B(x) \tag{1.5.3}$$

with suitable (4×4)-matrices G^j and B. This definition does not depend on coordinates and gauge, although the form of the matrices G^j and B clearly does. More precisely, under a change of coordinates $x^i \to \tilde{x}^i$ the operator (1.5.3) transforms into

$$i\left(G^k(\tilde{x})\,\frac{\partial \tilde{x}^j}{\partial x^k}\right)\frac{\partial}{\partial \tilde{x}^j} + B(\tilde{x})\,, \tag{1.5.4}$$

whereas a gauge transformation $\Psi \to U\Psi$ yields the operator

$$U\mathcal{D}U^{-1} \;=\; i\left(UG^jU^{-1}\right)\frac{\partial}{\partial x^j} + \left(UBU^{-1} + iUG^j(\partial_j U^{-1})\right). \tag{1.5.5}$$

We define the Dirac operator by the requirement that by choosing suitable coordinates and gauge, one can arrange that the matrices G^j in front of the partial derivatives "coincide locally" with the Dirac matrices of Minkowski space.

DEF. 1.5.1. *A differential operator $\mathcal{D}$ of first order is called* **Dirac operator** *if for every $p \in M$ there is a chart (x^i, U) around p and a gauge $(e_\alpha)_{\alpha=1,\ldots,4}$ such that $\mathcal{D}$ is of the form (1.5.3) with*

$$G^j(p) \;=\; \gamma^j\,, \tag{1.5.6}$$

where the γ^j are the Dirac matrices of Minkowski space in the Dirac representation (1.2.5).

It may seem unconventional that in this definition the zero order term B of the Dirac operator is not at all specified. Furthermore, our formulation as a gauge theory seems incomplete because we introduced local gauge transformations (1.5.2, 1.4.3), but not a corresponding gauge-covariant derivative (1.4.4). In order to clarify the situation, we shall now construct from the Dirac operator a gauge-covariant derivative D, also referred to as *spin derivative*. To this end, we must find matrices A_j which transform under local gauge transformations according to (1.4.5). This construction will also reveal the structure of the matrix B, and this will finally lead us to the definition of the so-called *physical Dirac operator*, which involves precisely the gravitational and electromagnetic fields.

In the chart and gauge where (1.5.6) holds, it is obvious from (1.2.3) that the anti-commutator of the matrices $G^j(p)$ gives the Minkowski metric. Using the transformation rules (1.5.4, 1.5.5), one sees that in a general coordinate system and gauge, their anti-commutator defines a Lorentz metric,

$$g^{jk}(x)\,\mathbb{1} \;=\; \frac{1}{2}\left\{G^j(x),\, G^k(x)\right\}. \tag{1.5.7}$$

In this way, the Dirac operator induces on the manifold a Lorentzian structure. We refer to the matrices G^j as the Dirac matrices in curved space-time. Since we can arrange that these matrices coincide locally with the Dirac matrices of Minkowski space, all manipulations of Dirac matrices can be performed at any given space-time point in an obvious way. In particular, the pseudoscalar matrix (1.2.13) now takes the more general form

$$\rho(x) \;=\; \frac{i}{4!}\,\varepsilon_{jklm}\,G^jG^kG^lG^m\,,$$

where the anti-symmetric tensor ε_{jklm} differs from the anti-symmetric symbol ϵ_{jklm} by the volume density, $\varepsilon_{jklm} = \sqrt{|\det g|}\,\epsilon_{jklm}$. The pseudoscalar matrix gives us again the notion of even and odd matrices and of chirality (1.2.14). Furthermore, we introduce the *bilinear matrices* σ^{jk} by

$$\sigma^{jk}(x) \;=\; \frac{i}{2}\,[G^j,\, G^k]\,.$$

As in Minkowski space, the matrices

$$G^j\,, \quad \rho G^j\,, \quad \mathbb{1}\,, \quad i\rho\,, \quad \sigma^{jk} \tag{1.5.8}$$

form a basis of the 16-dimensional (real) vector space of selfadjoint matrices (with respect to $\prec.|.\succ$). The matrices G^j and ρG^j are odd, whereas $\mathbb{1}$, $i\rho$ and σ^{jk} are even.

For the construction of the spin connection we must clearly consider derivatives. The Lorentzian metric (1.5.7) induces the Levi-Civita connection ∇, which defines the covariant derivative of tensor fields. Taking covariant derivatives of the Dirac matrices, $\nabla_k G^j = \partial_k G^j + \Gamma^j_{kl}\, G^l$, we obtain an expression which behaves under coordinate transformations like a tensor. However, it is not gauge covariant, because a gauge transformation (1.4.3) yields contributions involving first derivatives of U. More precisely, according to (1.5.5),

$$\begin{aligned} \nabla_k G^j \;\longrightarrow\; \nabla_k (U G^j U^{-1}) \;&=\; U(\nabla_k G^j) U^{-1} \,+\, (\partial_k U) G^j U^{-1} \,+\, U G^j (\partial_k U^{-1}) \\ &=\; U(\nabla_k G^j) U^{-1} \,-\, \big[U(\partial_k U^{-1}),\, U G^j U^{-1}\big]\,. \end{aligned} \tag{1.5.9}$$

We can use the second summand in (1.5.9) to partially fix the gauge.

LEMMA 1.5.2. *For every space-time point $p \in M$ there is a gauge such that*

$$\nabla_k G^j(p) \;=\; 0 \tag{1.5.10}$$

(for all indices j, k).

Proof. We start with an arbitrary gauge and construct the desired gauge with two subsequent gauge transformations:

(1) The matrix $\partial_j \rho$ is odd, because

$$0 \;=\; \partial_j \mathbb{1} \;=\; \partial_j(\rho\rho) \;=\; (\partial_j \rho)\rho + \rho(\partial_j \rho)\,.$$

As a consequence, the matrix $i\rho(\partial_j \rho)$ is selfadjoint. We can thus perform a gauge transformation U with $U(p) = \mathbb{1}$, $\partial_j U(p) = \frac{1}{2}\rho(\partial_j \rho)$. In the new gauge the matrix $\partial_j \rho(p)$ vanishes,

$$\partial_j \rho_{|p} \;\longrightarrow\; \partial_j (U \rho U^{-1})_{|p} \;=\; \partial_j \rho_{|p} + \frac{1}{2}\,[\rho(\partial_j \rho),\, \rho]_{|p} \;=\; \partial_j \rho_{|p} - \rho^2 (\partial_j \rho)_{|p} \;=\; 0\,.$$

Differentiating the relation $\{\rho, G^j\} = 0$, one sees that the matrix $\nabla_k G^j_{|p}$ is odd. We can thus represent it in the form

$$\nabla_k G^j_{|p} \;=\; \Lambda^j_{km}\, G^m_{|p} \,+\, \Theta^j_{km}\, \rho G^m \tag{1.5.11}$$

with suitable coefficients Λ^j_{km} and Θ^j_{km}.

This representation can be further simplified: According to Ricci's Lemma, $\nabla_n g^{jk} = 0$. Expressing the metric via the anti-commutation relations and differentiating through with the Leibniz rule, we obtain

$$\begin{aligned} 0 \;&=\; \{\nabla_n G^j,\, G^k\} \,+\, \{G^j,\, \nabla_n G^k\} \\ &=\; 2\Lambda^j_{nm}\, g^{mk} \,-\, \Theta^j_{nm}\, 2i\rho\sigma^{mk} \,+\, 2\Lambda^k_{nm}\, g^{mj} \,-\, \Theta^k_{nm}\, 2i\rho\sigma^{mj} \end{aligned} \tag{1.5.12}$$

and thus

$$\Lambda^j_{nm}\, g^{mk}_{|p} \;=\; -\Lambda^k_{nm}\, g^{mj}_{|p}\;. \tag{1.5.13}$$

In the case $j = k \neq m$, (1.5.12) yields that $\Theta^j_{nm} = 0$. For $j \neq k$, we obtain $\Theta^j_{nj}\,\sigma^{jk} + \Theta^k_{nk}\,\sigma^{kj} = 0$ and thus $\Theta^j_{nj} = \Theta^k_{nk}$ (j and k denote fixed indices, no summation is performed). We conclude that there are coefficients Θ_k with

$$\Theta^j_{km} \;=\; \Theta_k\, \delta^j_m\;. \tag{1.5.14}$$

(2) We perform a gauge transformation U with $U(p) = \mathbb{1}$ and

$$\partial_k U \;=\; -\frac{1}{2}\,\Theta_k\,\rho \;-\; \frac{i}{4}\,\Lambda^m_{kn}\, g^{nl}\,\sigma_{ml}\;.$$

Using the representation (1.5.11) together with (1.5.13, 1.5.14), the matrix $\nabla_k G^j$ transforms into

$$\begin{aligned}
\nabla_k G^j \;&\longrightarrow\; \nabla_k G^j \;+\; [\partial_k U,\, G^j] \\
&=\; \Lambda^j_{km}\, G^m \;+\; \Theta_k\,\rho G^j \;-\; \Theta_k\,\rho G^j \;-\; \frac{i}{4}\,\Lambda^m_{kn}\, g^{nl}\,[\sigma_{ml},\, G^j] \\
&=\; \Lambda^j_{km}\, G^m \;-\; \frac{i}{4}\,\Lambda^m_{kn}\, g^{nl}\, 2i\,(G_m\,\delta^j_l - G_l\,\delta^j_m) \\
&=\; \Lambda^j_{km}\, G^m \;+\; \frac{1}{2}\,\Lambda^m_{kn}\, g^{nj}\, G_m \;-\; \frac{1}{2}\,\Lambda^j_{km}\, G^m \;=\; 0\;.
\end{aligned}$$

■

We call a gauge satisfying condition (1.5.10) a *normal gauge* around p. In order to analyze the remaining gauge freedom, we let U be a transformation between two normal gauges. Then according to (1.5.9) and (1.5.10), the commutator $[U(\partial_k U^{-1}),\, UG^jU^{-1}]$ vanishes at p or, equivalently,

$$[i(\partial_k U^{-1})\, U,\, G^j]_{|p} \;=\; 0\;.$$

As is easily verified in the basis (1.5.8) using the commutation relations between the Dirac matrices, a matrix which commutes with all Dirac matrices is a multiple of the identity matrix. Moreover, the matrix $i(\partial_j U^{-1})\, U$ is selfadjoint because $(i(\partial_j U^{-1})U)^* = -iU^{-1}(\partial_j U) = -i\partial_j(U^{-1}U) + i(\partial_j U^{-1})U = i(\partial_j U^{-1})U$. We conclude that the matrix $i(\partial_j U^{-1})\, U$ is a real multiple of the identity matrix, and transforming it unitarily with U we see that it also coincides with the matrix $iU\,(\partial_j U^{-1})$. Under this strong constraint for the gauge transformation it is easy to find expressions with the required behavior (1.4.5) under gauge transformations. Namely, setting

$$a_j \;=\; \frac{1}{4}\,\mathrm{Re}\;\mathrm{Tr}(G_j\, B)\,\mathbb{1}\;, \tag{1.5.15}$$

where "Tr" denotes the trace of a 4×4-matrix, one sees from (1.5.5) that

$$a_j \;\longrightarrow\; a_j \;+\; \frac{1}{4}\,\mathrm{Re}\;\mathrm{Tr}\left(G_j G^k\, i(\partial_k U^{-1})\, U\right)\mathbb{1} \;=\; a_j + iU(\partial_j U^{-1})\;.$$

We can identify the a_j with the gauge potentials A_j and use (1.4.4) as the definition of the spin connection.

Def. 1.5.3. *The* **spin derivative** D *is defined by the condition that it behaves under gauge transformations (1.4.3) according to (1.4.6) and in normal gauges around p has the form*

$$D_j(p) \;=\; \frac{\partial}{\partial x^j} \;-\; ia_j \tag{1.5.16}$$

with the potentials a_j according to (1.5.15).

In general gauges, the spin derivative can be written as

$$D_j \;=\; \frac{\partial}{\partial x^j} \;-\; iE_j \;-\; ia_j \tag{1.5.17}$$

with additional matrices $E_j(x)$, which involve the Dirac matrices and their first derivatives. A short calculation shows that the trace of the matrix E_j does not change under gauge transformations, and since it vanishes in normal gauges, we conclude that the matrices E_j are trace-free. A straightforward calculation yields that they are explicitly given by

$$E_j \;=\; \frac{i}{2}\,\rho\,(\partial_j \rho) \;-\; \frac{i}{16}\,\mathrm{Tr}(G^m\,\nabla_j G^n)\,G_m G_n \;+\; \frac{i}{8}\,\mathrm{Tr}(\rho G_j\,\nabla_m G^m)\,\rho\,.$$

In the next two theorems we collect the basic properties of the spin connection.

THEOREM 1.5.4. *The spin derivative satisfies for all wave functions Ψ, Φ the equations*

$$[D_k, G^j] \;+\; \Gamma^j_{kl}\,G^l \;=\; 0 \tag{1.5.18}$$

$$\partial_j \prec\!\Psi \,|\, \Phi\!\succ \;=\; \prec\!D_j\Psi \,|\, \Phi\!\succ \;+\; \prec\!\Psi \,|\, D_j\Phi\!\succ\,. \tag{1.5.19}$$

Proof. The left side of (1.5.18) behaves under gauge transformations according to the adjoint representation $. \to U\,.\,U^{-1}$ of the gauge group. Thus it suffices to check (1.5.18) in a normal gauge, where

$$[D_k, G^j] + \Gamma^j_{kl}\,G^l \;=\; \nabla_k G^j \;-\; \frac{i}{4}\,\mathrm{Re}\;\mathrm{Tr}(G_j B)\,[\mathbb{1}, G^j] \;=\; 0\,.$$

Since both sides of (1.5.19) are gauge invariant, it again suffices to consider a normal gauge. The statement is then an immediate consequence of the Leibniz rule for partial derivatives and the fact that the spin derivative differs from the partial derivative by an imaginary multiple of the identity matrix (1.5.16). ∎

The identity (1.5.18) means that the coordinate and gauge invariant derivative of the Dirac matrices vanishes. The relation (1.5.19) shows that the spin connection is compatible with the spin scalar product.

We define *torsion* $\mathcal{T}$ and *curvature* $\mathcal{R}$ of the spin connection as the following 2-forms,

$$\mathcal{T}_{jk} \;=\; \frac{i}{2}\,([D_j, G_k] - [D_k, G_j])\;, \qquad\qquad \mathcal{R}_{jk} \;=\; \frac{i}{2}\,[D_j, D_k]\,.$$

THEOREM 1.5.5. *The spin connection is torsion-free. Curvature has the form*

$$\mathcal{R}_{jk} \;=\; \frac{1}{8}\,R_{mnjk}\,\sigma^{mn} \;+\; \frac{1}{2}\,(\partial_j a_k - \partial_k a_j) \tag{1.5.20}$$

where R_{mnjk} is the the Riemannian curvature tensor and the a_j are given by (1.5.15).

Proof. The identity (1.5.18) yields that

$$[D_j,\, G_k] \;=\; [D_j,\, g_{kl}\,G^l] \;=\; (\partial_j g_{kl})\,G^l \;-\; g_{kl}\,\Gamma^l_{jm}\,G^m \;=\; \Gamma^m_{jk}\,G_m$$

and thus, using that the Levi-Civita connection is torsion-free,

$$\mathcal{T}_{jk} \;=\; \frac{i}{2}\,(\Gamma^m_{jk} - \Gamma^m_{kj})\,G_m \;=\; 0\,.$$

Again using (1.5.18), we can rewrite the covariant derivative as a spin derivative,

$$G_l \, \nabla_k u^l \;=\; [D_k,\, G_l u^l] \,.$$

Iterating this relation, we can express the Riemann tensor (1.1.7) by

$$\begin{aligned} G_i \, R^i_{jkl} \, u^l &= [D_j,\, [D_k,\, G_l u^l]] - [D_k,\, [D_j,\, G_l u^l]] \\ &= [[D_j,\, D_k],\, G_l u^l] \;=\; -2i \, [\mathcal{R}_{jk},\, G_l u^l] \,. \end{aligned}$$

This equation determines curvature up to a multiple of the identity matrix,

$$\mathcal{R}_{jk}(x) \;=\; \frac{1}{8} \, R_{mnjk} \, \sigma^{mn} \;+\; \lambda_{jk} \mathbb{1} \,.$$

Thus it remains to compute the trace of curvature,

$$\frac{1}{4} \operatorname{Tr}(\mathcal{R}_{jk}) \, \mathbb{1} \;=\; \frac{1}{8} \operatorname{Tr}(\partial_j A_k - \partial_k A_j) \, \mathbb{1} \;=\; \frac{1}{2} \, (\partial_j a_k - \partial_k a_j) \,,$$

where we used (1.5.17) and the fact that the matrices E_j are trace-free. ∎

We come to the physical interpretation of the above construction. According to Lemma 1.5.2 we can choose a gauge around p such that the covariant derivatives of the Dirac matrices vanish at p. Moreover, choosing normal coordinates and making a global (=constant) gauge transformation, we can arrange that $G(p) = \gamma^j$ and $\partial_j g_{kl}(p) = 0$. Then the covariant derivatives at p reduce to partial derivatives, and we conclude that

$$G^j(p) \;=\; \gamma^j \,, \qquad \partial_k G^j(p) \;=\; 0 \,. \tag{1.5.21}$$

These equations have a large similarity with the conditions for normal coordinates (1.1.4), only the role of the metric is now played by the Dirac matrices. Indeed, differentiating (1.5.7) one sees that (1.5.21) implies (1.1.4). Therefore, (1.5.21) is a stronger condition which not only gives a constraint for the coordinates, but also for the gauge. We call a coordinate system and gauge where (1.5.21) is satisfied a *normal reference frame* around p.

In a normal reference frame, the Dirac matrices, and via (1.5.7) also the metric, are the same as in Minkowski space up to the order $o(x-p)$. According to the strong equivalence principle, the Dirac equation at p should coincide with that in Minkowski space. Now we use minimal coupling in the formulation (1.4.8) to conclude that there should be a normal gauge such that all gauge potentials vanish at p, and thus the Dirac operator at p should coincide with the free Dirac operator $i\partial\!\!\!/$. This physical argument allows us to specify the zero order term in (1.5.3).

DEF. 1.5.6. *A Dirac operator $\mathcal{D}$ is called* **physical Dirac operator** *if for any $p \in M$ there is a normal reference frame around p such that $B(p) = 0$.*

Equivalently, the physical Dirac operator could be defined as a differential operator of first order (1.5.3) with the additional structure that for any $p \in M$ there is a coordinate chart and gauge such that the following three conditions are satisfied,

$$G^j(p) = \gamma^j \,, \qquad \partial_k G^j(p) = 0 \,, \qquad B(p) = 0 \,.$$

This alternative definition has the disadvantage that it is a-priori not clear whether the second condition $\partial_k G^j(p) = 0$ can be satisfied for a general metric. This is the reason why we preferred to begin with only the first condition (Def. 1.5.1), then showed that

the second condition can be arranged by choosing suitable coordinates and gauge, and satisfied the third condition at the end (Def. 1.5.6).

In general coordinates and gauge, the physical Dirac operator can be written as

$$\mathcal{D} \;=\; iG^j D_j \;=\; iG^j \left(\partial_j - iE_j - ia_j\right),$$

where D is the spin connection of Def. 1.5.3. The matrices E_j take into account the gravitational field and are called *spin coefficients*, whereas the a_j can be identified with the *electromagnetic potential* (compare (1.2.7)). We point out that the gravitational field cannot be introduced into the Dirac equation by the simple replacement rule (1.4.7) because gravity has an effect on both the Dirac matrices and the spin coefficients. But factorizing the gauge group as $U(2,2) = U(1) \times SU(2,2)$, the $SU(2,2)$-gauge transformations are linked to the gravitational field because they influence G^j and E_j, whereas the $U(1)$ can be identified with the gauge group of electrodynamics. In this sense, we obtain a unified description of electrodynamics and general relativity as a $U(2,2)$ gauge theory. The Dirac equation

$$(\mathcal{D} - m)\, \Psi \;=\; 0$$

describes a Dirac particle in the gravitational and electromagnetic field. According to Theorem 1.5.5, the curvature of the spin connection involves both the Riemann tensor and the electromagnetic field tensor. We can write down the classical action in terms of these tensor fields, and variation yields the classical Einstein-Dirac-Maxwell equations.

For the probabilistic interpretation of the Dirac equation in curved space-time, we choose a space-like hypersurface $\mathcal{H}$ (corresponding to "space" for some observer) and consider in generalization of (1.2.17) on solutions of the Dirac equation the scalar product

$$(\Psi \,|\, \Phi) \;=\; \int_{\mathcal{H}} \prec\!\Psi \,|\, G^j \nu_j \, \Phi\!\succ d\mu_{\mathcal{H}}\,, \tag{1.5.22}$$

where ν is the future-directed normal on $\mathcal{H}$ and $d\mu_{\mathcal{H}}$ is the invariant measure on the Riemannian manifold $\mathcal{H}$. Then $(\Psi|\Psi)$ is the *normalization integral*, which we again normalize to one. Its integrand has the interpretation as the *probability density*. In analogy to (1.2.9) the *Dirac current* is introduced by $J^k = \prec\!\Psi \,|\, G^k \Psi\!\succ$. Using Theorem 1.5.4 one sees similar as in Minkowski space that the Dirac current is divergence-free, $\nabla_k J^k = 0$. From Gauss' theorem one obtains that the scalar product (1.5.22) does not depend on the choice of the hypersurface $\mathcal{H}$.

We finally remark that using Theorem 1.5.4 together with Gauss' theorem, one easily verifies that the physical Dirac operator is Hermitian with respect to the inner product

$$<\!\Psi \,|\, \Phi\!> \;:=\; \int_M \prec\!\Psi \,|\, \Phi\!\succ d\mu\,, \tag{1.5.23}$$

in which the wave functions (which need not satisfy the Dirac equation but must have a suitable decay at infinity) are integrated over the whole space-time. This inner product is not positive, but it will nevertheless play an important conceptual role in the next chapters.

CHAPTER 2

The Fermionic Projector in the Continuum

In the previous chapter we introduced the concept of the Dirac sea in order to give the negative-energy solutions of the free Dirac equation a physical meaning as anti-particle states (see §1.2, §1.3). Now we shall extend this concept to the case with interaction. We will see that the Dirac sea can still be introduced as a universal object in space-time, described mathematically by the so-called *fermionic projector* (§2.3). We develop the mathematical methods for an explicit analysis of the fermionic projector in position space (§2.5) and finally consider the normalization of the fermionic states (§2.6).

2.1. The External Field Problem

We begin with the simplest interaction: a classical external field in Minkowski space. In this situation the Dirac wave function is a solution of the Dirac equation

$$(i\partial\!\!\!/ + \mathcal{B} - m)\, \Psi \;=\; 0\,, \tag{2.1.1}$$

where the operator $\mathcal{B}$ is composed of the external potentials (as an example one may choose $\mathcal{B} = A\!\!\!/$ with A the electromagnetic potential (1.2.7)). In order to have current conservation (i.e. the identity $\partial_k J^k = 0$ with J according to (1.2.9)), we always assume that $\mathcal{B}$ is Hermitian (with respect to the spin scalar product). If $\mathcal{B}$ is *static* (=time independent), we can separate the time dependence of the wave function with a plane wave ansatz,

$$\Psi(t, \vec{x}) \;=\; e^{-i\omega t}\, \psi(\vec{x})\,. \tag{2.1.2}$$

The separation constant ω has the interpretation as the energy of the solution. Thus the energy is a conserved quantity, and its sign distinguishes between solutions of positive and negative energy. In more mathematical terms, for a static potential the Hamiltonian in (1.2.18) is time independent, and the sign of the spectrum of h gives a splitting of the solution space of the Dirac equation into the subspaces of positive and negative energy, respectively. As a consequence, our previous construction of the Dirac sea can be adapted: When building up the Fock space from the one-particle states, we cure the problem of the negative-energy solutions similar to (1.3.7–1.3.9) by redefining the vacuum and by formally exchanging the creation and annihilation operators corresponding to the negative-energy solutions.

The situation becomes much more difficult when $\mathcal{B}$ is *time-dependent*. In this case, the separation ansatz (2.1.2) no longer works. The energy is not conserved, and it is even possible that a solution which has positive energy at initial time will have negative energy at a later time. Expressed more mathematically, the Hamiltonian in (1.2.18) now depends explicitly on time, and therefore the sign of the spectrum of h no longer gives a canonical splitting of the solution space of the Dirac equation (this splitting would also depend on time). As a consequence, it is not clear which solutions have the interpretation as "negative-energy solutions" and thus correspond to anti-particle

states (1.3.8). For this reason, it is no longer obvious how to quantize the Dirac field in a canonical way. This difficulty is usually referred to as the *external field problem*. It becomes most evident in the setting of Klein's paradox, where one considers a step potential whose amplitude is larger than the mass gap (see [**BD1, T**]). However, we point out that the external field problem appears already for arbitrarily weak external fields, simply because the time dependence of $\mathcal{B}$ leads to a complicated mixing of the solutions of positive and negative energy. We could speak of a "solution of negative energy" only if it were a superposition of states which *all* had negative energy, and there seems no reason why such solutions should exist.

It is instructive to discuss the external field problem in the setting of pertubation theory. Consider a first order perturbation of the plane-wave solution $\Psi_{\vec{p}s\epsilon}$,

$$\Psi \;=\; \Psi_{\vec{p}s\epsilon} + \Delta\Psi + \mathcal{O}(\mathcal{B}^2)\,. \tag{2.1.3}$$

Substituting this ansatz into the Dirac equation (2.1.1), we obtain to first order in $\mathcal{B}$ the inhomogeneous Dirac equation

$$(i\partial\!\!\!/ - m)\,\Delta\Psi \;=\; -\mathcal{B}\,\Psi_{\vec{p}s\epsilon}\,. \tag{2.1.4}$$

If $s_m(x,y)$ is a *Green's function* of the free Dirac equation, characterized by the distributional equation

$$(i\partial\!\!\!/_x - m)\,s_m(x,y) \;=\; \delta^4(x-y)\,, \tag{2.1.5}$$

we can construct a solution of (2.1.4) by

$$\Delta\Psi \;=\; -\int d^4y\; s(x,y)\,\mathcal{B}(y)\,\Psi_{\vec{p}s\epsilon}(y) \tag{2.1.6}$$

(in order not to distract from the main ideas, we here calculate on a formal level; the analytic justification will be given at the end of §2.2). If the Green's function were unique, (2.1.3, 2.1.6) would give a unique procedure for perturbing the negative-energy solutions of the vacuum, making it possible to extend the notion of "negative-energy state" to the interacting theory (at least in first order perturbation theory).

The problem is that the Green's function is *not unique*, as we now briefly recall (for details see [**BD1**]). Taking the Fourier transform of (2.1.5),

$$s_m(x,y) \;=\; \int \frac{d^4k}{(2\pi)^4}\, s_m(k)\, e^{-ik(x-y)}\,, \tag{2.1.7}$$

we obtain the algebraic equation

$$(k\!\!\!/ - m)\,s_m(k) \;=\; \mathbb{1}\,.$$

Since the matrix $k\!\!\!/ - m$ is singular on the mass shell (see the argument after (1.2.19)), this equation can be solved for $s_m(k)$ only after using a $\pm i\varepsilon$-regularization on the mass shell. The most popular choices are the *advanced* and the *retarded* Green's functions defined by

$$s_m^\vee(k) \;=\; \lim_{\varepsilon \searrow 0} \frac{k\!\!\!/ + m}{k^2 - m^2 - i\varepsilon k^0} \qquad \text{and} \qquad s_m^\wedge(k) \;=\; \lim_{\varepsilon \searrow 0} \frac{k\!\!\!/ + m}{k^2 - m^2 + i\varepsilon k^0}\,, \tag{2.1.8}$$

respectively (with the limit $\varepsilon \searrow 0$ taken in the distributional sense). Computing their Fourier transform (2.1.7) with residues, one sees that they are *causal* in the sense that their supports lie in the upper and lower light cone, respectively,

$$\text{supp}\, s_m^\vee(x,.) \subset J_x^\vee\,, \qquad \text{supp}\, s_m^\wedge(x,.) \subset J_x^\wedge\,. \tag{2.1.9}$$

Another commmon choice is the *Feynman propagator*

$$s_m^F(k) \;:=\; \lim_{\varepsilon \searrow 0} \frac{\not{k} + m}{k^2 - m^2 + i\varepsilon}\,. \tag{2.1.10}$$

Taking the Fourier transform (2.1.7) with residues, one finds

$$s_m^F(x,y) \;=\; \frac{i}{(2\pi)^3} \int_{\mathbb{R}^3} (\not{k} + m)\, e^{-ik(x-y)}\Big|_{k=(\epsilon(t)\,\omega(\vec{p}),\,\vec{p})}\, d\mu_{\vec{p}} \tag{2.1.11}$$

with $t \equiv (y-x)^0$ and $\omega(\vec{p})$, $d\mu_{\vec{p}}$ as introduced after (1.3.2). Here ϵ denotes the step function $\epsilon(x) = 1$ for $x \geq 0$ and $\epsilon(x) = -1$ otherwise. Thus for positive t we get the integral over the upper mass shell, whereas for negative t we integrate over the lower mass shell. As a consequence, the Feynman propagator is not causal, but it is instead characterized by the *frequency conditions* that it is for positive and negative time t composed only of positive and negative frequencies, respectively. More systematically, the defining equation for the Green's function (2.1.5) determines s_m only up to a solution of the homogeneous Dirac equation. Thus we can write a general Green's function s_m in the form

$$s_m(x,y) \;=\; s_m^\vee(x,y) \;+\; a(x,y)\,, \tag{2.1.12}$$

where $a(x,y)$ is a linear combination of plane-wave solutions in the variable x, i.e.

$$a(x,y) \;=\; \sum_{s,\epsilon} \int_{\mathbb{R}^3} \Psi_{\vec{p}s\epsilon}(x)\, c_{\vec{p}s\epsilon}(y)\, d\mu_{\vec{p}}$$

with a suitable complex-valued function $c_{\vec{p}s\epsilon}(y)$.

Due to the non-uniqueness of the Green's function (2.1.12), the relations (2.1.3, 2.1.6) do *not* give a unique procedure for perturbing the plane-wave solutions $\Psi_{\vec{k}s\epsilon}$. In particular, it is not clear how to extend the notion of "negative energy state" to the interacting theory. This corresponds precisely to the external field problem. We conclude that in a perturbative approach, the external field problem becomes manifest in the non-uniqueness of the perturbation expansion.

Feynman [**Fe**] gave the frequency conditions in the Feynman propagator the physical interpretation as "positive-energy states moving to the future" and "negative-energy states moving to the past". Identifying "negative-energy states moving to the past" with "antiparticle states" he concluded that the Feynman propagator is distinguished from all other Green's functions in that it takes into account the particle/antiparticle interpretation of the Dirac equation in the physically correct way. Feynman proposed to perform the perturbation expansion exclusively with the Feynman propagator, thereby making the perturbation expansion unique. The flaw is that the frequency conditions are not invariant under general coordinate and gauge transformations (simply because such transformations "mix" positive and negative frequencies), and therefore Feynman's method is not compatible with the equivalence principle and the local gauge principle. This is not a problem for most calculations in physics, but it is not satisfying conceptually. Another approach is to work with the so-called *Hadamard states* [**H, Wa2**]. The disadvantage of this approach is that the states of the quantum field no longer have a particle interpretation. In other words, the notions of "particle" and "anti-particle" depend on the local observer, and therefore also the notion of the Dirac sea loses its universal meaning. We proceed in §2.2 by showing that the Dirac sea can indeed be introduced as a global object of space-time, even in the presence of a general interaction.

We finally remark that the above arguments apply in the same way for second quantized fields: we only need to replace $\mathcal{B}$ by an operator on a suitable bosonic Fock space. Also, our assumption that $\mathcal{B}$ is an *external* field is merely a technical simplification (more precisely, we disregard the dynamical equations for the bosonic fields, thereby also avoiding the divergences of QFT and the renormalization procedure), but it is not essential for our arguments. Namely, in a time-dependent interacting system, the Dirac wave functions satisfy (2.1.1), where $\mathcal{B}(t, \vec{x})$ is determined by the dynamical equations of the whole system. Solving these equations we can (at least in principle) compute $\mathcal{B}$, and applying our above arguments with this $\mathcal{B}$ as an external field, we conclude that the notion of "negative-energy state" ceases to exist. In what follows we will for simplicity again consider an external field, but we shall come back to coupled systems in §2.4.

2.2. The Causal Perturbation Expansion

We saw in the previous section that the external field problem for the Dirac equation (2.1.1) is equivalent to the non-uniqueness of the perturbation expansion for the individual states of the Dirac sea. We shall now solve this problem by considering the collection of all states of the Dirac sea. This will reveal an underlying causal structure, which will enable us to make the perturbation expansion unique[1]. We closely follow the constructions given in [**F4**].

In the vacuum, out of all plane-wave solutions of negative energy we form the object

$$P^{\text{sea}}(x,y) \;:=\; -\frac{m}{\pi} \sum_s \int_{\mathbb{R}^3} |\Psi_{\vec{p}s-} \succ\!\prec \Psi_{\vec{p}s-}|\; d\mu_{\vec{p}}\,. \tag{2.2.1}$$

Using the explicit form of the plane-wave solutions (1.2.23, 1.2.22, 1.2.20), we obtain the covariant formula

$$P^{\text{sea}}(x,y) \;=\; \int \frac{d^4k}{(2\pi)^4}\, (\not{k} + m)\, \delta(k^2 - m^2)\, \Theta(-k^0)\, e^{-ik(x-y)}\,, \tag{2.2.2}$$

which also shows that $P^{\text{sea}}(x,y)$ is a well-defined distribution. In order to get a connection to causality, we decompose $P^{\text{sea}}(x,y)$ in the form

$$P^{\text{sea}}(x,y) \;=\; \frac{1}{2}\,(p_m(x,y) \;-\; k_m(x,y))\,, \tag{2.2.3}$$

where p_m and k_m are the distributions

$$p_m(x,y) \;=\; \int \frac{d^4k}{(2\pi)^4}\, (\not{k} + m)\, \delta(k^2 - m^2)\, e^{-ik(x-y)} \tag{2.2.4}$$

$$k_m(x,y) \;=\; \int \frac{d^4k}{(2\pi)^4}\, (\not{k} + m)\, \delta(k^2 - m^2)\, \epsilon(k^0)\, e^{-ik(x-y)}\,. \tag{2.2.5}$$

[1] We remark for clarity that our "causal perturbation expansion" does not seem to be related to Scharf's "causal approach" to QED [**S**]. Scharf uses causality to avoid the ultraviolet divergences of perturbative QED, whereas in our setting of an external field all Feynman diagrams are finite anyway. On the other hand, Scharf is interested only in the scattering states, whereas our goal is to describe the dynamics also for intermediate times.

In order to relate the distribution k_m to the advanced and retarded Green's functions, we substitute the distributional equation

$$\lim_{\varepsilon \searrow 0} \left(\frac{1}{x - i\varepsilon} - \frac{1}{x + i\varepsilon} \right) = 2\pi i\, \delta(x)$$

into the formula for k_m in momentum space,

$$\begin{aligned} k_m(p) &= (\not{p} + m)\, \delta(p^2 - m^2)\, \epsilon(p^0) \\ &= \frac{1}{2\pi i}\, (\not{p} + m)\, \lim_{\varepsilon \searrow 0} \left[\frac{1}{p^2 - m^2 - i\varepsilon} - \frac{1}{p^2 - m^2 + i\varepsilon} \right] \epsilon(p^0) \\ &= \frac{1}{2\pi i}\, (\not{p} + m)\, \lim_{\varepsilon \searrow 0} \left[\frac{1}{p^2 - m^2 - i\varepsilon p^0} - \frac{1}{p^2 - m^2 + i\varepsilon p^0} \right]. \end{aligned} \tag{2.2.6}$$

Using (2.1.8) we get the simple formula

$$k_m = \frac{1}{2\pi i} \left(s_m^\vee - s_m^\wedge \right). \tag{2.2.7}$$

The support property (2.1.9) yields that k_m is causal in the sense that

$$\operatorname{supp} k_m(x, .) \subset J_x\,.$$

The distribution p_m is *not* causal, but it can be deduced from k_m as follows. For a diagonalizable matrix A with real eigenvalues we can uniquely define its *absolute value* $|A|$ as the diagonalizable matrix with non-negative eigenvalues and $|A|^2 = A^2$. The matrix $\not{k} + m$ in (2.2.4, 2.2.5) is diagonalizable with non-negative eigenvalues and thus

$$\left| \epsilon(k^0)\, (\not{k} + m) \right| = \not{k} + m\,. \tag{2.2.8}$$

Before applying this relation to (2.2.4, 2.2.5), it is useful to consider the above distributions P^{sea}, $p_m, \ldots$ as integral kernels of corresponding operators on the wave functions in space-time, for example

$$(P^{\text{sea}}\, \Psi)(x) := \int P^{\text{sea}}(x, y)\, \Psi(y)\, d^4y\,. \tag{2.2.9}$$

Then the operators k_m and p_m are diagonal in momentum space, and so (2.2.8) gives rise to the formal identity

$$p_m = |k_m|\,, \tag{2.2.10}$$

where $|\,.\,|$ now is the absolute value of an operator on wave functions in Minkowski space. With (2.2.3) and (2.2.7, 2.2.10) we have related the fermionic projector to the causal Green's functions in a way which can be generalized to the interacting theory, as we shall now make precise.

We begin with the perturbation expansion for the causal Green's functions. The retarded Green's function in the presence of the external field $\mathcal{B}$, denoted by $\tilde{s}_m^\wedge$, is characterized by the conditions

$$(i\partial\!\!\!/ + \mathcal{B} - m)\, \tilde{s}_m^\wedge(x, y) = \delta^4(x - y)\,, \qquad \operatorname{supp} \tilde{s}_m^\wedge(x, .) \subset J_x^\wedge\,. \tag{2.2.11}$$

The existence and uniqueness of the advanced Green's functions follows from the general theory of linear hyperbolic PDEs [**J, Ta**]. In short, for the existence proof one considers the solution of the Cauchy problem

$$(i\partial\!\!\!/ + \mathcal{B} - m)\, \Psi = f \in C_0^\infty((t_0, \infty) \times \mathbb{R}^3)^4\,, \qquad \Psi(t_0, \vec{x}) = 0\,;$$

by linearity it can be expressed as an integral over the inhomogeneity,

$$\Psi(x) \;=\; \int_{\mathbb{R}^4} \tilde{s}^\wedge_m(x,y)\, f(y)\, d^4y\,.$$

To prove uniqueness, one considers the difference of two retarded Green's functions $\tilde{s}^\wedge_{m,1}$ and $\tilde{s}^\wedge_{m,2}$,

$$\Psi(x) \;=\; \tilde{s}^\wedge_{m,1}(x,y) - \tilde{s}^\wedge_{m,2}(x,y)\,.$$

Then $\Psi(x)$ is for fixed y a solution of the homogeneous Dirac equation which vanishes identically on the half space $x^0 < y^0$. The uniqueness of the solution of the Cauchy problem yields that $\Psi \equiv 0$.

Expanding (2.2.11) in powers of $\mathcal{B}$, one obtains the perturbation series $\tilde{s}^\wedge_m = \sum_{n=0}^\infty s^\wedge_{(n)}$, where $s^\wedge_{(0)} = s^\wedge_m$ is the advanced Green's function of the vacuum, and the other summands are determined by the conditions that they are causal, $\operatorname{supp} s^\wedge_{(n)}(x,.) \in J^\wedge_x$, and satisfy the inductive relations

$$(i\partial\!\!\!/ - m)\, s^\wedge_{(n)} \;=\; -\mathcal{B}\, s^\wedge_{(n-1)} \qquad\qquad (n \geq 1).$$

Here we again used the operator notation (2.2.9) and considered $\mathcal{B}$ as a multiplication operator. The operator product

$$(-s^\wedge_m\, \mathcal{B}\, s^\wedge_m)(x,y) \;=\; -\int d^4z\; s^\wedge_m(x,z)\, \mathcal{B}(z)\, s^\wedge_m(z,y) \tag{2.2.12}$$

is causal in the sense that $\mathcal{B}(z)$ enters only for $z \in L^\wedge_x \cap L^\vee_y$ (the analytic justification of this and all other operator products in this section will be given in Lemma 2.2.2 below). In particular, the support of (2.2.12) is again in the past light cone. Furthermore, it satisfies the relation

$$(i\partial\!\!\!/ - m)\, (-s^\wedge_m\, \mathcal{B}\, s^\wedge_m) \;=\; -\mathcal{B}\, s^\wedge_m\,,$$

and can thus be identified with the operator $s^\wedge_{(1)}$. By iteration, we obtain for the other terms of the perturbation series the explicit formulas

$$s^\wedge_{(n)} \;=\; (-s^\wedge_m\, \mathcal{B})^n\, s^\wedge_m\,.$$

We conclude that the retarded Green's function can be represented as

$$\tilde{s}^\wedge_m \;=\; \sum_{k=0}^\infty \big(-s^\wedge_m\, \mathcal{B}\big)^k\, s^\wedge_m\,. \tag{2.2.13}$$

Similarly, we introduce the advanced Green's function $\tilde{s}^\vee_m$ by the conditions

$$(i\partial\!\!\!/ - m + \mathcal{B})\, \tilde{s}^\vee_m(x,y) \;=\; \delta^4(x-y)\,, \qquad \operatorname{supp} \tilde{s}^\vee_m(x,.) \;\subset\; J^\vee_x\,. \tag{2.2.14}$$

It has the perturbation expansion

$$\tilde{s}^\vee_m \;=\; \sum_{k=0}^\infty \big(-s^\vee_m\, \mathcal{B}\big)^k\, s^\vee_m\,. \tag{2.2.15}$$

Having uniquely introduced the causal Green's functions, we can now extend (2.2.7) to the case with interaction. Namely, we define the operator $\tilde{k}_m$ by

$$\tilde{k}_m \;=\; \frac{1}{2\pi i}\, \big(\tilde{s}^\vee_m - \tilde{s}^\wedge_m\big) \tag{2.2.16}$$

with the causal Green's functions as given by (2.2.13, 2.2.15). Finally, we also extend (2.2.10) to the case with interaction by setting

$$\tilde{p}_m \overset{\text{formally}}{:=} \sqrt{\tilde{k}_m^2}\,. \tag{2.2.17}$$

In the next theorem we will give this last relation a precise mathematical meaning and show that it gives rise to a unique perturbation expansion. It is most convenient to work with the Green's function

$$s_m := \frac{1}{2}(s_m^\vee + s_m^\wedge)\,. \tag{2.2.18}$$

Furthermore, we introduce the series of operator products

$$b_m^< = \sum_{k=0}^\infty (-s_m\,\mathcal{B})^k \quad , \qquad b_m = \sum_{k=0}^\infty (-\mathcal{B}\,s_m)^k\,\mathcal{B} \quad , \qquad b_m^> = \sum_{k=0}^\infty (-\mathcal{B}\,s_m)^k$$

and set for $Q \subset \mathbb{N}$

$$F_m(Q,n) = \begin{cases} p_m & \text{if } n \in Q \\ k_m & \text{if } n \notin Q \end{cases}\,.$$

THEOREM 2.2.1. *The relations (2.2.16, 2.2.17) uniquely determine the perturbation expansions for k_m and p_m. We have the explicit formulas*

$$\tilde{k}_m = \sum_{\beta=0}^\infty (-i\pi)^{2\beta}\, b_m^<\, k_m\, (b_m\, k_m)^{2\beta}\, b_m^> \tag{2.2.19}$$

$$\tilde{p}_m = \sum_{\beta=0}^\infty \sum_{\alpha=0}^{\left[\frac{\beta}{2}\right]} c(\alpha,\beta)\, G_m(\alpha,\beta) \tag{2.2.20}$$

with the coefficients

$$c(0,0) = 1 \tag{2.2.21}$$

$$c(\alpha,\beta) = \sum_{n=\alpha+1}^{\beta} (-1)^{n+1}\, \frac{(2n-3)!!}{n!\, 2^n} \binom{\beta-\alpha-1}{n-\alpha-1} \quad \text{for } \beta \geq 1 \tag{2.2.22}$$

and the operator products

$$G_m(\alpha,\beta) = \sum_{Q \in \mathcal{P}(\beta+1),\ \#Q = 2\alpha+1} (-i\pi)^{2\beta}$$
$$\times\, b_m^<\, F_m(Q,1)\, b_m k_m b_m\, F_m(Q,2)\, b_m k_m b_m \,\cdots\, b_m k_m b_m\, F_m(Q,\beta+1)\, b_m^>\,, \tag{2.2.23}$$

where $\mathcal{P}(n)$ denotes the set of subsets of $\{1,\ldots,n\}$ (we use the convention $l!! = 1$ for $l \leq 0$).

Proof. An explicit calculation using (2.1.5) shows that $(i\partial\!\!\!/ + \mathcal{B} - m)\, b_m^< = 0$. Since all operator products in (2.2.19) and (2.2.23) have a factor $b_m^<$ at the left, the operators $\tilde{p}_m$, $\tilde{k}_m$ are solutions of the Dirac equation,

$$(i\partial\!\!\!/ + \mathcal{B} - m)\, \tilde{p}_m = 0 = (i\partial\!\!\!/ + \mathcal{B} - m)\, \tilde{k}_m\,.$$

It remains to verify that the conditions (2.2.16, 2.2.17) are satisfied and to show uniqueness.

According to (2.2.7, 2.2.18), the advanced and retarded Green's function can be written as

$$s_m^\vee \;=\; s_m + i\pi\, k_m \quad , \qquad s_m^\wedge \;=\; s_m - i\pi\, k_m \,. \tag{2.2.24}$$

We substitute the series (2.2.13, 2.2.15) into (2.2.16),

$$\tilde{k}_m \;=\; \frac{1}{2\pi i} \sum_{k=0}^\infty \Big((-s_m^\vee\, \mathcal{B})^k\, s_m^\vee \;-\; (-s_m^\wedge\, \mathcal{B})^k\, s_m^\wedge \Big) \,, \tag{2.2.25}$$

insert (2.2.24) and expand. This gives a sum of operator products of the form

$$C_1\, \mathcal{B}\, C_2\, \mathcal{B} \,\cdots\, \mathcal{B}\, C_{l+1} \qquad \text{with} \qquad C_j = k_m \text{ or } C_j = s_m \,.$$

The contributions with an even number of factors k_m have the same sign for the advanced and retarded Green's functions and cancel in (2.2.25). The contributions with an odd number of k_m's occur in each Green's function exactly once and have the opposite sign. Using the notation

$$C_m(Q,n) \;=\; \begin{cases} k_m & \text{if } n \in Q \\ s_m & \text{if } n \notin Q \end{cases} \quad , \qquad Q \subset \mathbb{N} \,,$$

we can thus rewrite (2.2.25) in the form

$$\begin{aligned} \tilde{k}_m \;=\; \sum_{l=0}^\infty (-1)^l \sum_{Q \in \mathcal{P}(l+1),\; \#Q \text{ odd}} & (i\pi)^{\#Q-1} \\ & \times C_m(Q,1)\, \mathcal{B}\, C_m(Q,2)\, \mathcal{B} \cdots \mathcal{B}\, C_m(Q,l)\, \mathcal{B}\, C_m(Q,l+1) \,. \end{aligned}$$

After reordering the sums, this coincides with (2.2.19).

Next we want to give the relation (2.2.10) a mathematical meaning. To this end, we consider $m \geq 0$ as a variable mass parameter. Then we can form products of the operators p_m, k_m by manipulating the arguments of the distributions in momentum space. For example, using (2.2.4) we obtain

$$\begin{aligned} p_m(k)\, p_{m'}(k) \;&=\; (\not{k} + m)\, \delta(k^2 - m^2)\, (\not{k} + m')\, \delta(k^2 - (m')^2) \\ &=\; (k^2 + (m+m')\not{k} + mm')\, \delta(m^2 - (m')^2)\, \delta(k^2 - m^2) \\ &=\; (k^2 + (m+m')\not{k} + mm')\, \frac{1}{2m}\, \delta(m - m')\, \delta(k^2 - m^2) \\ &=\; \delta(m-m')\, p_m(k) \,, \end{aligned} \tag{2.2.26}$$

and similarly from (2.2.5),

$$p_m\, k_{m'} \;=\; k_{m'}\, p_m \;=\; \delta(m-m')\, k_m \tag{2.2.27}$$

$$k_m\, k_{m'} \;=\; \delta(m-m')\, p_m \,. \tag{2.2.28}$$

We remark that this formalism has some similarity with the bra/ket notation in quantum mechanics, if the position variable $\vec{x}$ is replaced by the mass parameter m. Equation (2.2.26) can be interpreted that the p_m are the spectral projectors of the free Dirac operator; the relations (2.2.27, 2.2.28) reflect the relative minus sign in k_m for the states on the upper and lower mass shell. In particular, one sees that $k_m\, k_{m'} = p_m\, p_{m'}$. This relation can be extended to the case with interaction,

$$\tilde{p}_m\, \tilde{p}_{m'} \;=\; \tilde{k}_m\, \tilde{k}_{m'} \,, \tag{2.2.29}$$

and gives a meaningful square of (2.2.10) (we will see in a moment that $\tilde{k}_m \tilde{k}_{m'}$ vanishes for $m \neq m'$). If our construction ensures that $\tilde{p}_m$ is a positive operator, (2.2.29) is even equivalent to (2.2.10).

Let us compute the product $\tilde{k}_m \tilde{k}_{m'}$ explicitly. The definitions (2.2.4, 2.2.5) and (2.2.18, 2.1.8) yield in analogy to (2.2.26) the formulas[2]

$$p_m\, s_{m'} \;=\; s_{m'}\, p_m \;=\; \mathrm{PP}\left(\frac{1}{m-m'}\right) p_m \tag{2.2.30}$$

$$k_m\, s_{m'} \;=\; s_{m'}\, k_m \;=\; \mathrm{PP}\left(\frac{1}{m-m'}\right) k_m \tag{2.2.31}$$

$$s_m\, s_{m'} \;=\; \mathrm{PP}\left(\frac{1}{m-m'}\right) (s_m - s_{m'})\,, \tag{2.2.32}$$

where $\mathrm{PP}(x^{-1}) = \frac{1}{2} \lim_{\varepsilon \searrow 0} [(x+i\varepsilon)^{-1} + (x-i\varepsilon)^{-1}]$ denotes the principal value. As a consequence, the operator products involving the factor $s_m \cdot s_{m'}$ are telescopic,

$$\sum_{p=0}^{n} k_m\, (\mathcal{B}\, s_m)^p\, (s_{m'}\, \mathcal{B})^{n-p}\, k_{m'} \;=\; 0 \qquad \text{for } n \geq 1. \tag{2.2.33}$$

This allows us to evaluate the following product,

$$k_m\, b_m^{>}\, b_{m'}^{<}\, k_{m'} \;=\; \delta(m-m')\, p_m\,. \tag{2.2.34}$$

With this formula, we can compute the square of (2.2.19),

$$\tilde{k}_m\, \tilde{k}_{m'} \;=\; \delta(m-m') \sum_{\beta_1,\beta_2=0}^{\infty} (-i\pi)^{2\beta_1+2\beta_2}\, b_m^{<}\, (k_m\, b_m)^{2\beta_1}\, p_m\, (b_m\, k_m)^{2\beta_2}\, b_m^{>}\,. \tag{2.2.35}$$

We could continue the proof by verifying explicitly that the product $\tilde{p}_m \tilde{p}_{m'}$ with $\tilde{p}_m$ according to (2.2.20) coincides with (2.2.35). This is a straightforward computation, but it is rather lengthy and not very instructive. We prefer to describe how the operator products (2.2.23) and the coefficients (2.2.22) can be derived. In order to make the proof more readable, we make the following simplifications. Since the factors $b_m^{<}$, $b_m^{>}$ cancel similar to (2.2.34) in telescopic sums, we can omit them in all formulas without changing the multiplication rules for the operator products. Then all operator products have k_m or p_m as their first and last factor, and we can multiply them with the rules (2.2.26–2.2.28). Since all these rules give a factor $\delta(m-m')$, we will in any case get the prefactor $\delta(m-m')$ as in (2.2.35). Therefore, we can just forget about all factors $\delta(m-m')$ and consider all expressions at the same value of m. Furthermore, we will omit the subscript '$_m$' and write the intermediate factors b as a dot '.'. After these simplifications, we end up with formal products of the form

$$F_1\,.\,F_2\,.\,F_3\,.\,\cdots\,.\,F_n \qquad \text{with} \qquad F_j = k \text{ or } F_j = p \tag{2.2.36}$$

and have the multiplication rules

$$p^2 \;=\; k^2 \;=\; 1 \quad , \qquad p\,k \;=\; k\,p \;=\; k\,. \tag{2.2.37}$$

[2] Online version: As noticed by A. Grotz, in (2.2.32) the summand $\pi^2 \delta(m-m')$ is missing. This error is corrected in the paper arXiv:0901.0334 [math-ph].

We must find a positive operator $\tilde{p}$ being a formal sum of operator products (2.2.36) such that

$$\tilde{p}^2 \;=\; \sum_{\beta_1,\beta_2=0}^{\infty} (-i\pi)^{2\beta_1+2\beta_2}\, (k\,.)^{2\beta_1}\, p\,(.\,k)^{2\beta_2}\,. \tag{2.2.38}$$

In this way, we have reduced our problem to the combinatorics of the operator products. As soon as we have found a solution $\tilde{p}$ of (2.2.38), the expression for $\tilde{p}_m$ is obtained by adding the subscripts '$_m$' and by inserting the factors $b_m^<$, b_m, $b_m^>$. Relation (2.2.29) follows as an immediate consequence of (2.2.38).

The basic step for the calculation of $\tilde{p}$ is to rewrite (2.2.38) in the form

$$\tilde{p}^2 \;=\; p + A \qquad \text{with} \qquad A \;=\; \sum_{(\beta_1,\beta_2)\neq(0,0)} (-i\pi)^{2\beta_1+2\beta_2}\, (k\,.)^{2\beta_1}\, p\,(.\,k)^{2\beta_2}\,. \tag{2.2.39}$$

The operator p is idempotent and acts as the identity on A, $Ap = pA = A$. Therefore, we can take the square root of $p + A$ with a formal Taylor expansion,

$$\tilde{p} \;=\; \sqrt{p+A} \;=\; p \;+\; \sum_{n=1}^{\infty} (-1)^{n+1}\, \frac{(2n-3)!!}{n!\, 2^n}\, A^n\,, \tag{2.2.40}$$

which uniquely defines $\tilde{p}$ as a positive operator.

It remains to calculate A^n. If we take the nth power of the sum in (2.2.39) and expand, we end up with one sum over more complicated operator products. We first consider how these operator products look like: The operator products in (2.2.39) all contain an even number of factors k and exactly one factor p. The factor p can be the 1st, 3rd,... factor of the product. Each combination of this type occurs in A exactly once. If we multiply n such terms, the resulting operator product consists of a total odd number of factors p, k. It may contain several factors p, which all occur at odd positions in the product. Furthermore, the total number of factors p is odd, as one sees inductively. We conclude that A^n consists of a sum of operator products of the form

$$(k\,.\,k\,.)^{q_1}\, p\,.\,k\,.\,(k\,.\,k\,.)^{q_2}\, p\,.\,k\,.\,(k\,.\,k\,.)^{q_3}\, \cdots\, (k\,.\,k\,.)^{q_{2\alpha+1}}\, p\,(.\,k\,.\,k)^{q_{2\alpha+2}} \tag{2.2.41}$$

with $\alpha, q_j \geq 0$. We set $\beta = 2\alpha + \sum_j q_j$. Notice that the number of factors p in (2.2.41) is $2\alpha+1$; the total number of factors p, k is $2\beta+1$. The form of the operator product gives the only restriction $0 \leq 2\alpha \leq \beta$ for the choice of the parameters α, β.

Next we count how often each operator product (2.2.41) occurs in the sum: The easiest way to realize (2.2.41) is to form the product of the $\alpha+1$ factors

$$\begin{aligned}&\big[(k.k.)^{q_1}\, p\,(.k.k)^{q_2+1}\big]\;\big[(k.k.)^{q_3+1}\, p\,(.k.k)^{q_4+1}\big]\\ &\qquad\qquad \cdots\;\big[(k.k.)^{q_{2\alpha+1}+1}\, p\,(.k.k)^{q_{2\alpha+2}}\big]\,.\end{aligned} \tag{2.2.42}$$

However, this is not the only way to factor (2.2.41). More precisely, to each factor in (2.2.42) we can apply the identities

$$\begin{aligned}(k\,.\,k\,.)^q\, p\,(.\,k\,.\,k)^r &= [(k\,.\,k\,.)^q\, p]\;[p\,(.\,k\,.\,k)^r]\\ (k\,.\,k\,.)^q\, p\,(.\,k\,.\,k)^r &= [(k\,.\,k\,.)^s\, p]\;\big[(k\,.\,k\,.)^{q-s}\, p\,(.\,k\,.\,k)^r\big]\\ (k\,.\,k\,.)^q\, p\,(.\,k\,.\,k)^r &= \big[(k\,.\,k\,.)^q\, p\,(.\,k\,.\,k)^{r-s}\big]\;[p\,(.\,k\,.\,k)^s]\,.\end{aligned}$$

By iteratively substituting these identities into (2.2.42), we can realize every factorization of (2.2.41). Each substitution step increases the number of factors by one. The steps are independent in the sense that we can fix at the beginning at which positions

in (2.2.42) the product shall be split up, and can then apply the steps in arbitrary order. There are $(\alpha+1)+(q_1-1)+\sum_{j=2}^{2\alpha+1} q_j + (q_{2\alpha+2}-1) = \beta-(\alpha+1)$ positions in (2.2.42) where we could split up the product (in the case $q_1=0$ or $q_{2\alpha+2}=0$, the counting of the positions is slightly different, but yields the same result). Since we want to have n factors at the end, we must choose $n-(\alpha+1)$ of these positions, which is only possible for $\alpha+1 \leq n \leq \beta$ and then gives $(\beta-\alpha-1)!/((n-\alpha-1)!\,(\beta-n)!)$ possibilities.

Combining these combinatorial factors with the constraints $0 \leq 2\alpha \leq \beta$ and $\alpha+1 \leq n \leq \beta$, we obtain for $n \geq 1$ the identity

$$\begin{aligned} A^n &= \sum_{\beta=n}^{\infty} \sum_{\alpha=0}^{\min\left(n-1,\left[\frac{\beta}{2}\right]\right)} \binom{\beta-\alpha-1}{n-\alpha-1} \sum_{Q \in \mathcal{P}(\beta+1),\ \#Q=2\alpha+1} \\ &\quad \times (-i\pi)^{2\beta}\, F(Q,1)\,.\,k\,.\,F(Q,2)\,.\,k\,.\,\cdots\,.\,k\,.\,F(Q,\beta+1) \end{aligned} \tag{2.2.43}$$

with $F(Q,n)=p$ for $n \in Q$ and $F(Q,n)=k$ otherwise. Notice that the last sum in (2.2.43) runs over all possible configurations of the factors p,k in the operator product (2.2.41) for fixed α,β. We finally substitute this formula into (2.2.40) and pull the sums over α,β outside. This gives the desired formula for $\tilde{p}$. ∎

We call the perturbation expansion of the above theorem the *causal perturbation expansion*. It allows us to define the Dirac sea in the presence of an external field canonically by

$$P^{\text{sea}}(x,y) \;=\; \frac{1}{2}\,(\tilde{p}_m - \tilde{k}_m)(x,y)\,.$$

In the next section the causal perturbation expansion will be extended to systems of Dirac seas, and in §2.4 we will discuss it in detail.

We conclude this section by showing that, under suitable regularity and decay assumptions on the external potentials, all operator products which appeared in this section are well-defined and finite.

LEMMA 2.2.2. *Let (C_j), $0 \leq j \leq n$, be a choice of operators $C_j \in \{k_m, p_m, s_m\}$. If the external potential $\mathcal{B}$ is smooth and decays so fast at infinity that the functions $\mathcal{B}(x)$, $x^i\mathcal{B}(x)$, and $x^ix^j\mathcal{B}(x)$ are integrable, then the operator product*

$$(C_n\,\mathcal{B}\,C_{n-1}\,\mathcal{B}\cdots\mathcal{B}\,C_0)(x,y) \tag{2.2.44}$$

is a well-defined tempered distribution on $\mathbb{R}^4 \times \mathbb{R}^4$.

Proof. Calculating the Fourier transform of (2.2.44) gives the formal expression

$$\begin{aligned} M(q_2,q_1) \;&:=\; \int \frac{d^4p_1}{(2\pi)^4} \cdots \int \frac{d^4p_{n-1}}{(2\pi)^4} C_n(q_2)\,\hat{\mathcal{B}}(q_1-p_{n-1}) \\ &\quad \times C_{n-1}(p_{n-1})\,\hat{\mathcal{B}}(p_{n-1}-p_{n-2})\;\cdots\;C_1(p_1)\,\hat{\mathcal{B}}(p_1-q_1)\,C_0(q_1)\,, \end{aligned} \tag{2.2.45}$$

where we consider the C_j as multiplication operators in momentum space and where $\hat{\mathcal{B}}$ denotes the Fourier transform of the function $\mathcal{B}$ (it is more convenient to work in momentum space because the operators C_j are then diagonal). We will show that $M(q_2,q_1)$ is a well-defined tempered distribution; the Lemma then immediately follows by transforming back to position space.

The assumptions on $\mathcal{B}$ yield that $\hat{\mathcal{B}}$ is C^2 and has rapid decay at infinity, i.e.

$$\sup_{q \in \mathbb{R}^4,\, |\kappa| \leq 2} |q^{i_1} \cdots q^{i_n}\, \partial_\kappa \hat{\mathcal{B}}(q)| \;<\; \infty$$

for all n, all tensor indices $i_1, \ldots, i_n$ and multi-indices κ (with $\kappa = (\kappa^1, \ldots, \kappa^q)$, $|\kappa| := q$). As is verified explicitly in momentum space, the distributions k_m, p_m or s_m are bounded in the Schwartz norms of the test functions involving derivatives of only first order, more precisely

$$|C(f)| \;\leq\; \text{const}\, \|f\|_{4,1} \qquad \text{with } C = k_m,\, p_m \text{ or } s_m \text{ and } f \in \mathcal{S},$$

where the Schwartz norms are as usual defined by

$$\|f\|_{p,q} \;=\; \max_{|I| \leq p,\, |J| \leq q} \sup_{x \in \mathbb{R}^4} |x^I\, \partial_J f(x)|\,.$$

As a consequence, we can apply the corresponding operators even to functions with rapid decay which are only C^1. Furthermore, we can form the convolution of such functions with C; this gives continuous functions (which will no longer have rapid decay, however). Since C involves first derivatives, a convolution decreases the order of differentiability of the function by one.

We consider the combination of multiplication and convolution

$$F(p_2) \;:=\; \int \frac{d^4 p_1}{(2\pi)^4}\, f(p_2 - p_1)\, C(p_1)\, g(p_1)\,, \tag{2.2.46}$$

where we assume that $f \in C^2$ has rapid decay and $g \in C^1$ is bounded together with its first derivatives, $\|g\|_{0,1} < \infty$. For any fixed p_2, the integral in (2.2.46) is well-defined and finite because $f(p_2 - .)\, g(.)$ is C^1 and has rapid decay. The resulting function F is C^1 and bounded together with its first derivatives, more precisely

$$\|F\|_{0,1} \quad \leq \quad \text{const}\, \|f\|_{4,2}\, \|g\|_{0,1}\,. \tag{2.2.47}$$

After these preparations, we can estimate the integrals in (2.2.45) from the right to the left: We choose two test functions $f, g \in \mathcal{S}(\mathbb{R}^4, \mathbb{C}^4)$ and introduce the functions

$$F_1(p_1) \;=\; \int \frac{d^4 q_2}{(2\pi)^4}\, \hat{\mathcal{B}}(p_1 - q_1)\, C_0(q_1)\, g(q_1) \tag{2.2.48}$$

$$F_j(p_j) \;=\; \int \frac{d^4 p_{j-1}}{(2\pi)^4}\, \hat{\mathcal{B}}(p_j - p_{j-1})\, C_{j-1}(p_{j-1})\, F_{j-1}(p_{j-1}) \quad,\quad 1 < j \leq n\,. \tag{2.2.49}$$

The integral (2.2.48) is of the form (2.2.46) and satisfies the above considered assumptions on the integrand. Using the bound (2.2.47), we can proceed inductively in (2.2.49). Finally, we perform the q_2-integration,

$$M(f, g) \;=\; \int \frac{d^4 q_2}{(2\pi)^4}\, f(q_2)\, C_n(q_2)\, F_n(q_2)\,.$$

We conclude that M is a linear functional on $\mathcal{S}(\mathbb{R}^4, \mathbb{C}^4) \times \mathcal{S}(\mathbb{R}^4, \mathbb{C}^4)$, which is bounded in the Schwartz norm $\|.\|_{4,1}$ of the test functions. ∎

Using the language of quantum field theory, we also refer to the summands of the perturbation expansions as *Feynman diagrams*. Then the result of the last lemma can be understood from the fact that in an external field one only encounters tree diagrams, which are all finite. Clearly, the existence of the perturbation expansion

to every order does not imply the convergence of the perturbation series, and we will come back to this problem in §2.5.

2.3. Definition of the Fermionic Projector

In this section we introduce the mathematical framework for describing a many-fermion system in the presence of an external field. To this end, we first extend the construction of §2.2 to a system of Dirac seas of in general different masses, which may involve chiral massless Dirac seas. Then we introduce particles and anti-particles by occupying additional states and creating "holes" in the Dirac seas, respectively. Our construction is intended to be so general that it allows us to model the fermion configuration of the standard model (see §5.1). For clarity, we postpone the question of how the fermionic states are to be normalized to §2.6.

First, we need to introduce a distribution $P^{\text{sea}}(x,y)$ which describes the system in the vacuum. The most general ansatz is to take a direct sum of sums of Dirac seas,

$$P^{\text{sea}} \;=\; \bigoplus_{a=1}^{N} \sum_{\alpha=1}^{g(a)} P^{\text{sea}}_{a\alpha}\,, \tag{2.3.1}$$

where $g(a)$ are positive integers and the summands $P^{\text{sea}}_{a\alpha}$ are Dirac seas of a form similar to (2.2.2). The direct sum increases the total number of components of the wave functions, the so-called *spin dimension*, to $4N$. The direct summands are called sectors, and we refer to the indices a and α as the *sector* and *generation index*, respectively. For each Dirac sea we introduce a mass parameter $m_{n\alpha} \geq 0$. In order to allow for chiral massless Dirac seas, we introduce (4×4)-matrices $X_{a\alpha}$ with

$$X_{a\alpha} \;=\; \begin{cases} \mathbb{1} & \text{if } m_{a\alpha} > 0 \\ \mathbb{1}, \chi_L \text{ or } \chi_R & \text{if } m_{a\alpha} = 0 \end{cases}$$

and set

$$P^{\text{sea}}_{a\alpha} \;=\; \frac{1}{2}\, X_{a\alpha}\, (p_{m_{a\alpha}} - k_{m_{a\alpha}})\,. \tag{2.3.2}$$

We refer to P^{sea} as defined by (2.3.1, 2.3.2) as the *fermionic projector of the vacuum*. It is sometimes useful to consider P^{sea} as a matrix in the sectors indices,

$$(P^{\text{sea}})^a_b \;=\; \delta^a_b\, \frac{1}{2} \sum_{\alpha=1}^{g(a)} X_{a\alpha}\, (p_{m_{a\alpha}} - k_{m_{a\alpha}})$$

with $a, b = 1, \ldots, N$.

Since each sector may involve several Dirac seas of different masses, it seems impossible to write the fermionic projector of the vacuum as a solution of a suitable Dirac equation, and thus we have no starting point for a perturbation expansion. In order to bypass this problem, we replace the sum in (2.3.1) by a direct sum and introduce the so-called *auxiliary fermionic projector* by

$$P^{\text{sea}} \;=\; \bigoplus_{a=1}^{N} \bigoplus_{\alpha=1}^{g(a)} P^{\text{sea}}_{a\alpha}\,. \tag{2.3.3}$$

Using the same notation as for the fermionic projector is usually no problem because it will be clear from the context whether the fermionic projector or the auxiliary

fermionic projector is meant. In case of potential confusion we write the auxiliary fermionic projector as a matrix in the sector and generation indices,

$$(P^{\text{sea}})^{(a\alpha)}_{(b\beta)} \;=\; \delta^a_b\, \delta^\alpha_\beta \frac{1}{2}\, X_{a\alpha}\, (p_{m_{a\alpha}} - k_{m_{a\alpha}})$$

with $a, b = 1, \ldots, N$, $\alpha = 1, \ldots, g(a)$, $\beta = 1, \ldots, g(b)$). In this notation, one also sees that the fermionic projector can be obtained from the auxiliary fermionic projector by taking the so-called *partial trace* over the generations,

$$(P^{\text{sea}})^a_b \;=\; \sum_{\alpha=1}^{g(a)} \sum_{\beta=1}^{g(b)} (P^{\text{sea}})^{(a\alpha)}_{(b\beta)}\,. \tag{2.3.4}$$

We introduce the operators

$$p \;=\; \bigoplus_{a=1}^{N} \bigoplus_{\alpha=1}^{g(a)} p_{m_{a\alpha}}\,, \qquad\qquad k \;=\; \bigoplus_{a=1}^{N} \bigoplus_{\alpha=1}^{g(a)} k_{m_{a\alpha}}$$

and define the matrices

$$X \;=\; \bigoplus_{a=1}^{N} \bigoplus_{\alpha=1}^{g(a)} X_{a\alpha}\,, \qquad\qquad Y \;=\; \frac{1}{m} \bigoplus_{a=1}^{N} \bigoplus_{\alpha=1}^{g(a)} m_{a\alpha}\,,$$

which are called *chiral asymmetry matrix* and *mass matrix*, respectively. Here m is an arbitrary mass parameter; a convenient choice is $m = \max_{a,\alpha} m_{a\alpha}$. These operators act on direct sums of Dirac wave functions, i.e. on functions of the form $\Psi = (\Psi^{(a\alpha)}(x))$ with $\Psi^{(a\alpha)}$ a 4-component Dirac spinor. On these wave funtions, we introduce the *spin scalar product* by

$$\prec \Psi \,|\, \Phi \succ (x) \;=\; \sum_{a=1}^{n} \sum_{\alpha=1}^{g(a)} \prec \Psi^{(a\alpha)} \,|\, \Phi^{(a\alpha)} \succ_{\text{Dirac}}\,, \tag{2.3.5}$$

where $\prec .|. \succ_{\text{Dirac}}$ is the usual spin scalar product on Dirac spinors (1.2.8). In generalization of (1.5.23) we also introduce the indefinite inner product

$$< \Psi \,|\, \Phi > \;=\; \int_M \prec \Psi \,|\, \Phi \succ \, d\mu\,. \tag{2.3.6}$$

Then the operators p and k are Hermitian with respect to $<.|.>$, and the mass matrix Y is Hermitian with respect to the spin scalar product. Using the above notation, we can write the auxiliary fermionic projector as

$$P^{\text{sea}}(x,y) \;=\; X\, \frac{1}{2}\, (p(x,y) - k(x,y))\,. \tag{2.3.7}$$

Since $m_l = 0$ for $X_l \neq \mathbb{1}$ and since the operators $p_{m=0}$, $k_{m=0}$ are odd, we have alternatively

$$P^{\text{sea}}(x,y) \;=\; \frac{1}{2}\, (p(x,y) - k(x,y))\, X^*\,, \tag{2.3.8}$$

where X^* is the adjoint with respect to the spin scalar product. The auxiliary fermionic projector is a solution of the free Dirac equation

$$(i\partial\!\!\!/_x - mY)\, P^{\text{sea}}(x,y) \;=\; 0\,. \tag{2.3.9}$$

Our strategy is to extend the definition of the auxiliary fermionic projector to the interacting system and then to get back to the fermionic projector by taking the partial trace (2.3.4).

In order to describe the system of Dirac seas in the presence of an external field, we insert a differential operator $\mathcal{B} = (\mathcal{B}^{(a\alpha)}_{(b\beta)})$ into the Dirac equation (2.3.9),

$$(i\partial\!\!\!/_x + \mathcal{B} - mY)\, P^{\text{sea}}(x,y) \;=\; 0\,. \tag{2.3.10}$$

We always assume that $\mathcal{B}$ is Hermitian with respect to the inner product $<.|.>$. The causal perturbation expansion for the operators k and p can be carried out exactly as in §2.2: We define the advanced and retarded Green's functions by

$$s^\vee \;=\; \bigoplus_{a=1}^{N} \bigoplus_{\alpha=1}^{g(a)} s^\vee_{m_{a\alpha}}\,, \qquad\qquad s^\wedge \;=\; \bigoplus_{a=1}^{N} \bigoplus_{\alpha=1}^{g(a)} s^\wedge_{m_{a\alpha}}\,.$$

Their perturbation expansion is, in analogy to (2.2.13, 2.2.15), uniquely given by

$$\tilde{s}^\vee \;=\; \sum_{k=0}^{\infty} (-s^\vee\, \mathcal{B})^k\, s^\vee\,, \qquad\qquad \tilde{s}^\wedge \;=\; \sum_{k=0}^{\infty} (-s^\wedge\, \mathcal{B})^k\, s^\wedge\,. \tag{2.3.11}$$

The method of Theorem 2.2.1 now yields the following result.

THEOREM 2.3.1. *The perturbation expansion for p and k is uniquely determined by the conditions*

$$\tilde{k} \;=\; \frac{1}{2\pi i}\,(\tilde{s}^\vee - \tilde{s}^\wedge)\,, \qquad\qquad \tilde{p} \;\overset{\text{formally}}{=}\; \sqrt{\tilde{k}^2}\,. \tag{2.3.12}$$

We have the explicit formulas

$$\tilde{k} \;=\; \sum_{\beta=0}^{\infty} (-i\pi)^{2\beta}\; b^< \,k\, (b\,k)^{2\beta}\, b^>\,, \qquad\qquad \tilde{p} \;=\; \sum_{\beta=0}^{\infty} \sum_{\alpha=0}^{\left[\frac{\beta}{2}\right]} c(\alpha,\beta)\, G(\alpha,\beta)$$

with

$$\begin{aligned} c(0,0) \;&=\; 1\,, \\ c(\alpha,\beta) \;&=\; \sum_{n=\alpha+1}^{\beta} (-1)^{n+1}\, \frac{(2n-3)!!}{n!\, 2^n} \binom{\beta-\alpha-1}{n-\alpha-1} \qquad \text{for } \beta \geq 1 \end{aligned}$$

and

$$\begin{aligned} G(f,g) \;=\; & \sum_{Q \in \mathcal{P}(\beta+1),\; \#Q = 2\alpha+1} (-i\pi)^{2\beta} \\ & \times b^<\, F(Q,1)\, bkb\, F(Q,2)\, bkb \,\cdots\, bkb\, F(Q,\beta+1)\, b^>\,, \end{aligned}$$

where $\mathcal{P}(n)$ is the set of subsets of $\{1,\ldots,n\}$ and where we use the notation

$$\begin{aligned} s \;&=\; \frac{1}{2}\,(s^\vee + s^\wedge)\,, \qquad F(Q,n) \;=\; \begin{cases} p & \text{if } n \in Q \\ k & \text{if } n \notin Q \end{cases} \\ b^< \;&=\; \sum_{k=0}^{\infty} (-s\,\mathcal{B})^k\,, \qquad b \;=\; \sum_{k=0}^{\infty} (-\mathcal{B}\,s)^k\,\mathcal{B}\,, \qquad b^> \;=\; \sum_{k=0}^{\infty} (-\mathcal{B}\,s)^k\,. \end{aligned}$$

The contributions to this perturbation expansion are all well-defined according to Lemma 2.2.2.

After this straightforward generalization, we come to the more subtle question of how to define P^{sea} when a chiral asymmetry is present. The obvious idea is to set in generalization of (2.3.7)

$$P^{\text{sea}}(x,y) \;=\; X\,\frac{1}{2}\,(\tilde{p}-\tilde{k})(x,y)\,. \tag{2.3.13}$$

This is not convincing, however, because we could just as well have defined $P^{\text{sea}}(x,y)$ in analogy to (2.3.8) by $P^{\text{sea}} = \frac{1}{2}(\tilde{p}-\tilde{k})\,X^*$, which does not coincide with (2.3.13) as soon as X, X^* do not commute with $\mathcal{B}$. Actually, this arbitrariness in defining the Dirac sea reflects a basic problem of the causal perturbation expansion for systems with chiral asymmetry. In order to describe the problem in more detail, we consider the perturbation calculation for k to first order. According to (2.3.11, 2.3.12),

$$\begin{aligned}\tilde{k} \;&=\; k \;-\; \frac{1}{2\pi i}\,(s^\vee\,\mathcal{B}\,s^\vee \;-\; s^\wedge\,\mathcal{B}\,s^\wedge) \;+\; \mathcal{O}(\mathcal{B}^2) \\ &=\; k \;-\; s\,\mathcal{B}\,k \;-\; k\,\mathcal{B}\,s \;+\; \mathcal{O}(\mathcal{B}^2)\,.\end{aligned} \tag{2.3.14}$$

This expansion is causal in the sense that $\tilde{k}(x,y)$ only depends on $\mathcal{B}$ in the "diamond" $(L_x^\vee \cap L_y^\wedge) \cup (L_y^\vee \cap L_x^\wedge)$, as is obvious from (2.3.14). It is not clear, however, how to insert the chiral asymmetry matrix into this formula. It seems most natural to replace all factors k by Xk,

$$(\tilde{Xk}) \;=\; Xk \;-\; s\,\mathcal{B}\,Xk \;-\; Xk\,\mathcal{B}\,s \;+\; \mathcal{O}(\mathcal{B}^2)\,. \tag{2.3.15}$$

Unfortunately, this expression cannot be written similar to (2.3.14) with the advanced and retarded Green's functions, which means that the causality of the expansion is in general lost. In order to avoid this problem, one might want to insert X at every factor s, k,

$$\begin{aligned}(\tilde{Xk}) \;&=\; Xk \;-\; Xs\,\mathcal{B}\,Xk \;-\; Xk\,\mathcal{B}\,Xs \;+\; \mathcal{O}(\mathcal{B}^2) \\ &=\; Xk \;-\; \frac{1}{2\pi i}\,(Xs^\vee\,\mathcal{B}\,Xs^\vee \;-\; Xs^\wedge\,\mathcal{B}\,Xs^\wedge) \;+\; \mathcal{O}(\mathcal{B}^2)\,.\end{aligned} \tag{2.3.16}$$

This expansion is causal similar to (2.3.14). In general, however, it does not satisfy the Dirac equation $(i\partial\!\!\!/ + \mathcal{B} - m)\,(\tilde{Xk}) = 0$, which does not seem to be what we want.

The only way to resolve this problem is to impose that the perturbation expansions (2.3.15) and (2.3.16) should coincide. This yields a condition for the operator $\mathcal{B}$, which can be characterized as follows. We demand that

$$Xs^\vee\,\mathcal{B}\,Xs^\vee \;=\; s^\vee\,\mathcal{B}\,Xs^\vee \;=\; Xs^\vee\,\mathcal{B}\,s^\vee\,. \tag{2.3.17}$$

Since the operator $s^\vee_{m=0}$ is odd, we have $Xs^\vee = s^\vee X^*$. Substituting into the second equation of (2.3.17) yields the condition $X^*\,\mathcal{B} \;=\; \mathcal{B}\,X$. Since X is idempotent, this condition automatically implies the first equation of (2.3.17). We formulate the derived condition for the whole Dirac operator $i\partial\!\!\!/ + \mathcal{B} - mY$ and thus combine it with the fact that chiral fermions are massless (i.e. $X^*Y = YX = Y$) and that X is composed of chiral projectors (which implies that $X^*\partial\!\!\!/ = \partial\!\!\!/ X$).

DEF. 2.3.2. *The Dirac operator is called* **causality compatible** *with X if*

$$X^*\,(i\partial\!\!\!/ + \mathcal{B} - mY) \;=\; (i\partial\!\!\!/ + \mathcal{B} - mY)\,X\,. \tag{2.3.18}$$

In the perturbation expansion to higher order, the condition (2.3.18) allows us to commute X through all operator products. Using idempotence $X^2 = X$, we can moreover add factors X to the product; in particular,

$$X\, C_1 \mathcal{B} C_1 \mathcal{B} \cdots \mathcal{B} C_n \;=\; X C_1 \mathcal{B} X C_1 \mathcal{B} \cdots \mathcal{B} X C_n \qquad \text{with} \qquad C_j = p,\, C_j = k \text{ or } C_j = s\,.$$

This ensures that the perturbation expansion is also well-defined to higher order. For a Dirac operator which is causality compatible with X, the *auxiliary fermionic projector* is defined canonically by (2.3.13).

So far the auxiliary fermionic projector describes a system of Dirac seas in the presence of an external field. In order to insert particles and anti-particles into the system, we add the projectors on particle states and substract the projectors on anti-particle states,

$$\begin{aligned} P(x,y) \;&=\; P^{\text{sea}}(x,y) \\ &\quad + c_{\text{norm}} \sum_{k=1}^{n_f} |\Psi_k(x) \succ\!\prec \Psi_k(y)| \;-\; c_{\text{norm}} \sum_{l=1}^{n_a} |\Phi_l(x) \succ\!\prec \Phi_l(y)|\,, \end{aligned} \tag{2.3.19}$$

where Ψ_k and Φ_l are an orthogonal set of solutions of the Dirac equation, and the Φ_l must lie in the image of P^{sea} (for the normalization constant c_{norm} see §2.6). The parameters n_f and n_a denote the total number of particles and anti-particles, respectively. We usually avoid the issue of convergence of the sums in (2.3.19) by assuming that $n_f, n_a < \infty$, but one could clearly also consider an infinite number of particles and/or anti-particles. Finally, the *fermionic projector* is obtained from this expression by taking similar to (2.3.4) the partial trace,

$$(P)_b^a \;=\; \sum_{\alpha=1}^{g(a)} \sum_{\beta=1}^{g(b)} (P)_{(b\beta)}^{(a\alpha)}\,. \tag{2.3.20}$$

Theorem 2.3.1 together with (2.3.13, 2.3.19, 2.3.20) yields a mathematical framework for describing a general many-fermion system in the presence of an external field. Our construction makes Dirac's concept of a "sea of interacting particles" mathematically precise. Apart from the causality compatibility condition (2.3.18) and the regularity conditions in Lemma 2.2.2, the operator $\mathcal{B}$ is completely arbitrary. We point out that we do not use the fermionic Fock space formalism of canonical quantum field theory; the connection to this formalism will be explained in §3.2 and Appendix A.

2.4. Interpretation and Consequences

With the definition of the fermionic projector we radically departed from the usual concept of the Dirac sea as "all negative-energy solutions" of the Dirac equation. Instead, we used causality in a particular way. In order to clarify the connection between our definition and the usual concept of the Dirac sea, we now describe how the above constructions simplify in the special situation that $\mathcal{B}$ is *static*. If considered as multiplication operators, static potentials map functions of positive and negative frequency into functions of positive and negative frequency, respectively. Since the operators p, k and s are diagonal in momentum space, they clearly preserve the sign of the frequency too. Thus

$$[\Pi^{\pm}, p] \;=\; [\Pi^{\pm}, k] \;=\; [\Pi^{\pm}, s] \;=\; [\Pi^{\pm}, \mathcal{B}] \;=\; 0\,, \tag{2.4.1}$$

where the operators $\Pi^\pm$ are the projectors onto the states of positive and negative frequency, respectively (i.e. in momentum space, $\Pi^\pm$ are the operators of multiplication by the functions $\Theta(\pm k^0)$). The operators p and k differ only by a relative minus sign for the states of positive and negative frequency,

$$\Pi^\pm p \;=\; \pm\, \Pi^\pm k\,.$$

Using this relation together with (2.4.1), we can replace pairs of factors p by pairs of factors k. For example,

$$\begin{aligned} \cdots p\,\mathcal{B} \cdots p\,\mathcal{B} \cdots &= \cdots p\,\mathcal{B} \cdots p\,\mathcal{B} \cdots\, (\Pi^+ + \Pi^-) \\ &= \Pi^+ (\cdots k\,\mathcal{B} \cdots k\,\mathcal{B} \cdots) \;+\; \Pi^- (\cdots (-k)\,\mathcal{B} \cdots (-k)\,\mathcal{B} \cdots) \\ &= \cdots k\,\mathcal{B} \cdots k\,\mathcal{B} \cdots\,, \end{aligned} \tag{2.4.2}$$

where the dots '$\cdots$' denote any combination of the operators s, k, p and $\mathcal{B}$. This allows us to simplify the formula for $\tilde{p}$ by using only one factor p in every operator product. After going through the details of the combinatorics, one obtains the formula

$$\tilde{p} \;=\; \sum_{b=0}^{\infty} (-i\pi)^{2b}\, b^< \, p\, (b\,k)^{2b}\, b^>\,.$$

Thus the fermionic projector (2.3.13) can be written as

$$P^{\text{sea}}(x,y) \;=\; \sum_{b=0}^{\infty} (-i\pi)^{2b}\, b^< \, \left[\frac{1}{2}\, X\, (p-k)\right]\, (b\,k)^{2b}\, b^>\,.$$

This equation shows that P^{sea} is composed of all negative-frequency eigenstates of the Dirac operator (notice that the expression in the brackets $[\cdots]$ is the fermionic projector of the vacuum and that all other factors preserve the sign of the frequency). We conclude that for static potentials our definition reduces to the usual concept of the Dirac sea as "all negative-energy states."

In order to get a better understanding of the time-dependent situation, we next consider a *scattering process*. For simplicity, we consider a system of one Dirac sea and assume that the scattering takes place in finite time $t_0 < t < t_1$. This means that the wave functions Ψ satisfy the Dirac equation (2.1.1) with $\mathcal{B}$ supported in a finite time interval,

$$\mathcal{B}(t,\vec{x}) \;=\; 0 \qquad \text{if } t \notin [t_0, t_1]\,. \tag{2.4.3}$$

As a consequence, $\Psi(t,\vec{x})$ is for $t < t_0$ a solution of the free Dirac equation. We uniquely extend this free solution to the whole Minkowski space and denote it by Ψ_{in},

$$(i\partial\!\!\!/ - m)\, \Psi_{\text{in}} \;=\; 0 \qquad \text{and} \qquad \Psi_{\text{in}}(t,\vec{x}) \;=\; \Psi(t,\vec{x}) \quad \text{for } t < t_0.$$

Similarly, $\Psi(t,\vec{x})$ is also for $t > t_1$ a solution of the free Dirac equation; we denote its extension by Ψ_{out},

$$(i\partial\!\!\!/ - m)\, \Psi_{\text{out}} \;=\; 0 \qquad \text{and} \qquad \Psi_{\text{out}}(t,\vec{x}) \;=\; \Psi(t,\vec{x}) \quad \text{for } t > t_1.$$

The wave functions Ψ_{in} and Ψ_{out} are called the incoming and outgoing *scattering states*, respectively. Recall that the dynamics of the wave functions is described infinitesimally by the Dirac equation in the Hamiltonian form (1.2.18), where h is a symmetric operator on the Hilbert space $(\mathcal{H}, (.,.))$ with scalar product (1.2.17). Integrating this equation from t_0 to t_1, we obtain a unitary operator S which maps the incoming scattering states to the corresponding outgoing states,

$$\Psi_{\text{out}} \;=\; S\, \Psi_{\text{in}}\,. \tag{2.4.4}$$

The operator S is called scattering operator or *S-matrix*.

Using the scattering states, we can introduce fermionic projectors which describe the vacua in the asymptotic past and future: For an observer in the past $t < t_0$, the external potential is zero. Thus it is natural for him to describe the vacuum with the free Dirac sea (2.2.1). If this Dirac sea is extended to the whole Minkowski space with external potential, one gets the object

$$P^\wedge(x,y) \;=\; -\frac{m}{\pi} \sum_s \int_{\mathbb{R}^3} |\Psi^\wedge_{\vec{p}s-} \succ\!\prec \Psi^\wedge_{\vec{p}s-}|\; d\mu_{\vec{p}}\,, \tag{2.4.5}$$

where the wave functions $\Psi^\wedge_{\vec{p}s\epsilon}$ are the solutions of the Dirac equation (2.1.1) whose incoming scattering states are the plane wave solutions $\Psi_{\vec{p}s\epsilon}$,

$$(i\partial\!\!\!/ + \mathcal{B} - m)\, \Psi^\wedge_{\vec{p}s\epsilon} \;=\; 0 \qquad \text{and} \qquad (\Psi^\wedge_{\vec{p}s\epsilon})_{\text{in}} \;=\; \Psi_{\vec{p}s\epsilon}\,.$$

Using the support conditions (2.4.3, 2.1.9), we can express the state $\Psi^\wedge_{\vec{p}s\epsilon}$ in a perturbation series,

$$\Psi^\wedge_{\vec{p}s\epsilon} \;=\; \sum_{n=0}^\infty (-s^\wedge\, \mathcal{B})^n\, \Psi_{\vec{p}s\epsilon}\,.$$

Substituting this formula into (2.4.5) we obtain for $P^\wedge$ a perturbation expansion which involves only the retarded Green's functions,

$$P^\wedge \;=\; \sum_{n_1,n_2=0}^\infty (-s^\wedge\, \mathcal{B})^{n_1}\, P^{\text{vac}}\, (-\mathcal{B}\, s^\wedge)^{n_2}\,, \tag{2.4.6}$$

where P^{vac} stands for the free Dirac sea (2.2.1). Accordingly, an observer in the future $t > t_0$ describes the vacuum by the fermionic projector

$$P^\vee(x,y) \;=\; -\frac{m}{\pi} \sum_s \int_{\mathbb{R}^3} |\Psi^\vee_{\vec{p}s-} \succ\!\prec \Psi^\vee_{\vec{p}s-}|\; d\mu_{\vec{p}}\,, \tag{2.4.7}$$

where

$$(i\partial\!\!\!/ + \mathcal{B} - m)\, \Psi^\vee_{\vec{p}s\epsilon} \;=\; 0 \qquad \text{and} \qquad (\Psi^\vee_{\vec{p}s\epsilon})_{\text{out}} \;=\; \Psi_{\vec{p}s\epsilon}\,.$$

Its perturbation expansion involves only the advanced Green's function,

$$P^\vee \;=\; \sum_{n_1,n_2=0}^\infty (-s^\vee\, \mathcal{B})^{n_1}\, P^{\text{vac}}\, (-\mathcal{B}\, s^\vee)^{n_2}\,. \tag{2.4.8}$$

Using (2.4.4) in (2.4.5, 2.4.7), we can describe the fermionic projectors in the asymptotic past and future with the S-matrix by

$$P^\wedge_{\text{in}} \;=\; P^{\text{vac}} \;=\; P^\vee_{\text{out}}\,, \qquad P^\wedge_{\text{out}} \;=\; S P^{\text{vac}} S^{-1}\,, \qquad P^\vee_{\text{in}} \;=\; S^{-1} P^{\text{vac}} S\,. \tag{2.4.9}$$

What makes the scattering process interesting is the fact that the vacua in the asymptotic past and future in general do not coincide. Consider for example the physical system described by the fermionic projector $P := P^\wedge$. For the observer in the past, the system is in the vacuum. However, if $P^\wedge \neq P^\vee$, the system will *not* be in the vacuum for the observer in the future. This means that for him, positive frequency states are occupied and negative frequency states are unoccupied and thus the system

contains particles and anti-particles. More precisely, if we write the fermionic projector in analogy to (2.3.19) as

$$\begin{aligned} P(x,y) \;&=\; P^\vee(x,y) \\ &\quad + c_{\text{norm}} \sum_{k=1}^{n_f} |\Psi_k(x) \succ\!\prec \Psi_k(y)| \;-\; c_{\text{norm}} \sum_{l=1}^{n_a} |\Phi_l(x) \succ\!\prec \Phi_l(y)| \,, \end{aligned} \tag{2.4.10}$$

then the Ψ_k and Φ_l are the wave functions of the particles and anti-particles, respectively. These particles and anti-particles are physical reality; the observer in the future can detect them by making suitable experiments. This is the physical effect of *pair creation*. Using (2.4.9) one can express the pair creation completely in terms of the S-matrix. Other scattering processes are described similarly.

We point out that describing the scattering process with the two observers in the past and future is merely a matter of convenience. The physical process can be described equivalently (although maybe less conveniently) in the reference frame of any other observer. To give a concrete example, we consider an observer in the future who is in a reference frame moving with constant acceleration. This leads to the so-called *Unruh effect* , which we now briefly outline (for details see e.g. [**Wa2**]). For the accelerated observer, space-time is stationary (i.e. his time direction is a Killing field, but it is not a unit normal to the hypersurfaces $t = \text{const}$), and this allows him to use the separation ansatz (2.1.2) with t his proper time. The sign of ω gives him a splitting of the solution space into solutions of positive and negative energy. Using Dirac's hole interpretation corresponding to this splitting, he finds for the many-fermion system described by P an infinite number of particles and anti-particles in a thermal equilibrium. This bizarre effect shows that the interpretation of the physical system in terms of particles and anti-particles does depend on the observer. Nevertheless, the Unruh effect does not contradict the pair creation experiments made by the future observer at rest. Namely, if the accelerated observer wants to explain the experiments by the future observer at rest, he must take into account that he himself is feeling a gravitational field, and that for him the experimental apparatus used by the observer at rest is in accelerated motion. It turns out that these additional effects just compensate the Unruh effect, so that the predictions by the accelerated observer are in complete agreement with the observations of the particles and anti-particles in (2.4.10) by the future observer at rest. More generally, all quantities which can be measured in experiments can be expressed in terms of the S-matrix. Since the S-matrix does not depend on the particle/anti-particle interpretation, it is clear that all experiments can be explained equivalently in any reference frame.

We just saw that the particle/anti-particle interpretation of a fermionic system may depend on the observer. Actually, the situation is even worse for an observer in the time period $t_0 < t < t_1$ when the interaction takes place. For him, the system is neither static nor stationary. Therefore, he has no notion of "negative-energy state", and thus for him the particle/anti-particle interpretation completely breaks down. Taking into account that a scattering process is an idealized process and that in a real physical situation there will be no region of space-time where no interaction takes place, we come to the disillusioning conclusion that for a local observer under generic conditions, a many-fermion system has no interpretation in terms of particles and anti-particles.

The causal perturbation expansion yields a canonical object P^{sea} which describes the Dirac sea in the scattering process, even in the region with interaction $t_0 < t < t_1$.

Its construction is explicitly covariant and independent of a local observer. Decomposing the fermionic projector in the form (2.3.19), we obtain a canonical interpretation of the many-fermion system in terms of particles and anti-particles. One should keep in mind that P^{sea} does not correspond to the vacuum of any local observer, but is a global object of space-time. As a consequence, also the particle/anti-particle interpretation in (2.3.19) can be associated only to an abstract "global observer" in space-time. More specifically, comparing Theorem 2.3.1 and (2.3.13) with (2.4.6, 2.4.8), one sees that P^{sea} coincides neither with $P^\wedge$ nor with $P^\vee$. Since its perturbation expansion involves both retarded and advanced Green's functions, it can be considered as being some kind of "interpolation" between $P^\wedge$ and $P^\vee$.

Let us now discuss our assumption on the potential $\mathcal{B}$ as being *external*. As explained at the end of §2.1, this is no restriction in principle because one can first solve the physical equations of the coupled system and then can define P^{sea} for the external potential $\mathcal{B}$ as given by the solution of the coupled system. Clearly, this procedure cannot be carried out in practice, but this is of no relevance for the theoretical considerations here. The important point is that P^{sea} is not defined locally; for its definition we need to know $\mathcal{B}$ in the whole space-time. This is puzzling because the conventional physical equations are local and causal, and this is the first time that an object appears which is defined in a non-local way. One might conclude that P^{sea} is an object which is not compatible with causality and should therefore have no physical significance. Our concept is the opposite: We regard the appearance of a non-local object as a first hint that locality and causality should be given up in the strict sense. In order to formulate physical equations which could replace the conventional local and causal equations, we shall consider the fermionic projector as the fundamental object in space-time.

Before we can make these ideas precise in Chapter 3, we need to analyze the fermionic projector in the continuum in more detail. One task is to understand what "causality" of the causal perturbation expansion means precisely. At the moment, we know that causality was used for the definition of P^{sea}, but that nevertheless the fermionic projector is a nonlocal and non-causal object. We need to find out how these seemingly contradicting facts fit together. Also, we must understand better how P^{sea} depends on $\mathcal{B}$. More specifically, we need to analyze what information on the external potentials is encoded in P^{sea}, and how this information can be extracted. Finally, we must specify how the fermionic states in (2.3.19) are to be normalized. The next sections provide the mathematical tools for answering these questions.

2.5. The Light-Cone Expansion

The light-cone expansion is a very useful technique for analyzing the fermionic projector near the light cone. In order to give a brief but self-contained introduction, we will explain the methods and results of [**F6**] leaving out many proofs and technical details. Our setting is that of §2.3 with several sectors and generations (2.3.1). It suffices to consider the *auxiliary fermionic projector* because the fermionic projector is obtained from it simply by taking the partial trace (2.3.20). We again assume that the Dirac operator in (2.3.10) is causality compatible (2.3.18) and that the operator $\mathcal{B}$ is Hermitian with respect to the inner product (2.3.6). Furthermore, we assume as in [**F6**] that $\mathcal{B}$ is a multiplication operator composed of *chiral* and *scalar/pseudoscalar potentials*,

$$\mathcal{B}(x) \;=\; \chi_L \,\A_R(x) \;+\; \chi_R \,\A_L(x) \;+\; \Phi(x) \;+\; i\rho\,\Xi(x)\,. \tag{2.5.1}$$

We note for clarity that these potentials may act non-trivially on the sectors and generations (e.g. writing the right-handed potential as a matrix, $A_R = (A_R)^{(a\alpha)}_{(b\beta)}$, the matrix elements are independent vector fields with the only constraint that the matrix must be Hermitian). In particular, the potentials in (2.5.1) do in general not commute, and thus we must in products be careful with the order of multiplication. For an operator $\mathcal{B}$ which involves bilinear and gravitational potentials see [**F5**].

DEF. 2.5.1. *A distribution $A(x,y)$ on $\mathbb{R}^4 \times \mathbb{R}^4$ is of the order $\mathcal{O}((y-x)^{2p})$, $p \in \mathbb{Z}$, if the product*

$$(y-x)^{-2p}\, A(x,y)$$

is a regular distribution (=a locally integrable function). It has the **light-cone expansion**

$$A(x,y) \;=\; \sum_{j=g}^{\infty} A^{[j]}(x,y) \tag{2.5.2}$$

with $g \in \mathbb{Z}$ if the distributions $A^{[j]}(x,y)$ are of the order $\mathcal{O}((y-x)^{2j})$ and if A is approximated by the partial sums in the sense that for all $p \geq g$,

$$A(x,y) \;-\; \sum_{j=g}^{p} A^{[j]}(x,y) \qquad \text{is of the order } \mathcal{O}((y-x)^{2p+2})\,. \tag{2.5.3}$$

The light-cone expansion describes the behavior of a distribution near the light cone. More precisely, the expansion parameter $(y-x)^2$ vanishes if y lies on the light cone centered at x, and thus the distributions $A^{[j]}(x,y)$ approximate $A(x,y)$ for y in a neighborhood of L_x. The first summand $A^{[g]}(x,y)$ gives the leading order of $A(x,y)$ on the light cone, $A^{[g+1]}$ gives the next order on the light cone, etc. If the distribution A is singular on the light cone, the parameter g will be negative. Note that the distributions $A^{[j]}$ are determined only up to contributions of higher order $\mathcal{O}((y-x)^{2j+2})$, but this ambiguity will not lead to any problems in what follows. We point out that we do not demand that the infinite series in (2.5.2) converges. This series is defined only via the approximation by the partial sums (2.5.3). Despite this formal character of the series, the light-cone expansion completely describes the behavior of $A(x,y)$ near the light cone. This situation can be seen in analogy to the Taylor expansion of a smooth, nonanalytic function. Although the Taylor series does in general not converge, the Taylor polynomials give local approximations of the function. An important difference to a Taylor expansion is that the $A^{[j]}(x,y)$ approximate $A(x,y)$ even for points x and y which are far apart. We only need that y is close to the light cone L_x, which is an unbounded hypersurface in $\mathbb{R}^4$. In this sense, the light-cone expansion is a *non-local expansion*.

For clarity, we begin with the light-cone expansion for the causal Green's functions, and we will later extend the results to the fermionic projector. In order to get a first idea of how the light-cone expansion can be carried out, we consider the free advanced Green's function $s^\vee_m$ as defined by (2.1.7, 2.1.8). We can pull the Dirac matrices out of the Fourier integral by setting

$$s^\vee_m(x,y) \;=\; (i\partial\!\!\!/_x + m)\, S^\vee_{m^2}(x,y)\,, \tag{2.5.4}$$

where $S^\vee_{m^2}$ is the advanced Green's function of the Klein-Gordon operator,

$$S^\vee_{m^2}(x,y) \;=\; \lim_{\varepsilon \searrow 0} \int \frac{d^4k}{(2\pi)^4}\, \frac{1}{k^2 - m^2 - i\varepsilon k^0}\, e^{-ik(x-y)}\,. \tag{2.5.5}$$

This Fourier integral can be computed explicitly; we expand the resulting Bessel function in a power series,

$$\begin{aligned} S^\vee_{m^2}(x,y) &= -\frac{1}{2\pi}\,\delta(\xi^2)\,\Theta(\xi^0) + \frac{m^2}{4\pi}\,\frac{J_1(\sqrt{m^2\xi^2})}{\sqrt{m^2\xi^2}}\,\Theta(\xi^2)\,\Theta(\xi^0) \\ &= -\frac{1}{2\pi}\,\delta(\xi^2)\,\Theta(\xi^0) + \frac{m^2}{8\pi}\sum_{j=0}^\infty \frac{(-1)^j}{j!\,(j+1)!}\,\frac{(m^2\xi^2)^j}{4^j}\,\Theta(\xi^2)\,\Theta(\xi^0)\,, \end{aligned} \tag{2.5.6}$$

where we used the abbreviation $\xi \equiv y-x$. This calculation shows that $S^\vee_{m^2}(x,y)$ has a $\delta((y-x)^2)$-like singularity on the light cone. Furthermore, one sees that $S^\vee_{m^2}$ is a power series in m^2. The important point for us is that the contributions of higher order in m^2 contain more factors $(y-x)^2$ and are thus of higher order on the light cone. More precisely,

$$\left(\frac{d}{dm^2}\right)^n S^\vee_{m^2\,|m^2=0}(x,y) \qquad \text{is of the order} \quad \mathcal{O}((y-x)^{2n-2})\,. \tag{2.5.7}$$

According to (2.5.4), the Dirac Green's function is obtained by taking the first partial derivatives of (2.5.6). Thus $s^\vee_m(x,y)$ has a singularity on the light cone which is even $\sim \delta'((y-x)^2)$. The higher order contributions in m are again of increasing order on the light cone. This means that we can view the Taylor expansion of (2.5.4) in m,

$$s^\vee_m(x,y) = \sum_{n=0}^\infty \frac{m^{2n}}{n!}\,(i\partial\!\!\!/ + m)\left(\frac{d}{dm^2}\right)^n S^\vee_{m^2\,|\,m^2=0}(x,y)\,, \tag{2.5.8}$$

as a light-cone expansion of the Green's function.

Writing the light-cone expansion of $s^\vee_m$ in the form (2.5.8) clearly is more convenient than working with the explicit formula (2.5.6). This is our motivation for using an expansion with respect to the mass parameter also in the presence of the external field. Expanding the perturbation expansion (2.3.11) in m gives a double series in powers of m and $\mathcal{B}$. In order to combine these two expansions in a single perturbation series, we write the mass matrix and the scalar/pseudoscalar potentials together by setting

$$Y_L(x) = Y - \frac{1}{m}\,(\Phi(x) + i\Xi(x))\,, \qquad Y_R(x) = Y - \frac{1}{m}\,(\Phi(x) - i\Xi(x))\,. \tag{2.5.9}$$

The matrices $Y_{L/R}(x)$ are called *dynamical mass matrices*; notice that $Y_L^* = Y_R$. With this notation, we can rewrite the Dirac operator as

$$i\partial\!\!\!/ + \mathcal{B} - mY = i\partial\!\!\!/ + B \qquad \text{with} \tag{2.5.10}$$

$$B = \chi_L\,(A\!\!\!/_R - m\,Y_R) + \chi_R\,(A\!\!\!/_L - m\,Y_L)\,. \tag{2.5.11}$$

For the light-cone expansion of the Green's functions, we shall always consider B as the perturbation of the Dirac operator. This has the advantage that the free theory consists of zero-mass fermions, and thus the Green's functions of the free Dirac operator have the simple form

$$s^\vee(x,y) = i\partial\!\!\!/_x\, S^\vee_{m^2=0}(x,y)\,, \qquad s^\wedge(x,y) = i\partial\!\!\!/_x\, S^\wedge_{m^2=0}(x,y)\,. \tag{2.5.12}$$

The Green's functions with interaction are given in analogy to (2.3.11) by the perturbation series

$$\tilde{s}^\vee = \sum_{k=0}^\infty (-s^\vee\,B)^k s^\vee\,, \qquad \tilde{s}^\wedge = \sum_{k=0}^\infty (-s^\wedge\,B)^k s^\wedge\,. \tag{2.5.13}$$

We remark that this perturbation expansion around zero mass is most convenient, but not essential for the light-cone expansion; see [**F5**] for a light cone expansion of a massive Dirac sea.

Our first goal is to perform the light-cone expansion of each Feynman diagram in (2.5.13). Using an inductive construction based on Lemma 2.5.2 below, this will give us the result of Theorem 2.5.3. Since the construction is exactly the same for the advanced and retarded Green's functions, we will omit all superscripts '$\vee$' and '$\wedge$'. The formulas for the advanced and retarded Green's functions are obtained by adding either superscripts '$\vee$' or '$\wedge$' to all operators s and S. For the mass expansion of the operator S_{m^2}, we set $a = m^2$ and introduce the notation

$$S^{(l)} \;=\; \left(\frac{d}{da}\right)^l S_{a|a=0} \qquad\qquad (l \geq 0). \tag{2.5.14}$$

Let us derive some computation rules for the $S^{(l)}$. S_a satisfies the defining equation of a Klein-Gordon Green's function

$$(-\Box_x - a)\, S_a(x,y) \;=\; \delta^4(x-y)\,.$$

Differentiating with respect to a yields

$$-\Box_x S^{(l)}(x,y) \;=\; \delta_{l,0}\, \delta^4(x-y) \;+\; l\, S^{(l-1)}(x,y) \qquad\qquad (l \geq 0). \tag{2.5.15}$$

For $l = 0$, this formula does not seem to make sense because $S^{(-1)}$ is undefined. However, the expression is meaningful if one keeps in mind that in this case the factor l is zero, and thus the whole second summand vanishes. We will also use this convention in the following calculations. Next, we differentiate the formula for S_a in momentum space,

$$S_a^\vee(p) \;=\; \frac{1}{p^2 - a - i\varepsilon p^0}\,, \qquad\qquad S_a^\wedge(p) \;=\; \frac{1}{p^2 - a + i\varepsilon p^0}\,, \tag{2.5.16}$$

with respect to both p and a. Comparing the results gives the relation

$$\frac{\partial}{\partial p^k} S_a(p) \;=\; -2p_k\, \frac{d}{da} S_a(p)\,,$$

or, after expanding in the parameter a,

$$\frac{\partial}{\partial p^k} S^{(l)}(p) \;=\; -2p_k\, S^{(l+1)}(p) \qquad\qquad (l \geq 0). \tag{2.5.17}$$

This formula also determines the derivatives of $S^{(l)}$ in position space. Namely,

$$\begin{aligned}
\frac{\partial}{\partial x^k} S^{(l)}(x,y) \;&=\; \int \frac{d^4p}{(2\pi)^4}\, S^{(l)}(p)\, (-ip_k)\, e^{-ip(x-y)} \\
&\overset{(2.5.17)}{=}\; \frac{i}{2} \int \frac{d^4p}{(2\pi)^4}\, \frac{\partial}{\partial p^k} S^{(l-1)}(p)\, e^{-ip(x-y)} \\
&=\; -\frac{i}{2} \int \frac{d^4p}{(2\pi)^4}\, S^{(l-1)}(p)\, \frac{\partial}{\partial p^k} e^{-ip(x-y)} \\
&=\; \frac{1}{2}\, (y-x)_k\, S^{(l-1)}(x,y) \qquad\qquad (l \geq 1).
\end{aligned} \tag{2.5.18}$$

We iterate this relation to compute the Laplacian,

$$\begin{aligned} -\Box_x S^{(l)}(x,y) &= -\frac{1}{2}\,\frac{\partial}{\partial x^k}\Big((y-x)^k\, S^{(l-1)}(x,y)\Big) \\ &= 2\, S^{(l-1)}(x,y) \;-\; \frac{1}{4}\,(y-x)^2\, S^{(l-2)}(x,y) \qquad (l \geq 2). \end{aligned}$$

After comparing with (2.5.15), we conclude that

$$(y-x)^2\, S^{(l)}(x,y) \;=\; -4l\, S^{(l+1)}(x,y) \qquad (l \geq 0). \tag{2.5.19}$$

Furthermore, $S^{(l)}(x,y)$ is only a function of $(y-x)$, and thus

$$\frac{\partial}{\partial x^k} S^{(l)}(x,y) \;=\; -\frac{\partial}{\partial y^k} S^{(l)}(x,y) \qquad (l \geq 0). \tag{2.5.20}$$

Finally, it is convenient to use the identity (2.5.18) also in the case $l = 0$ and to use it as the definition of S^{-1},

$$\frac{\partial}{\partial x^k} S^{(l)}(x,y) \;=\; \frac{1}{2}\,(y-x)_k\, S^{(l-1)}(x,y) \qquad (l \geq 0). \tag{2.5.21}$$

Notice that $S^{(-1)}$ itself remains undefined, only the combination $(y-x)_k S^{(-1)}$ is given a mathematical meaning as the partial derivative of the distribution $2S^{(0)}$.

The next lemma gives the light-cone expansion of an operator product where a potential V is sandwiched between two mass-derivatives of the Green's function. This expansion is the key for the subsequent iterative light-cone expansion of all Feynman diagrams. We always assume without saying that the potentials satisfy the regularity conditions of Lemma 2.2.2.

LEMMA 2.5.2. *The operator product $S^{(l)}\, V\, S^{(r)}$ with $l, r \geq 0$ has the light-cone expansion*

$$\begin{aligned} (S^{(l)}\, V\, S^{(r)})(x,y) &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^1 \alpha^l\, (1-\alpha)^r\, (\alpha-\alpha^2)^n \\ &\qquad \times (\Box^n V)_{|\alpha y + (1-\alpha)x}\, d\alpha\; S^{(n+l+r+1)}(x,y)\,. \end{aligned} \tag{2.5.22}$$

The fact that line integrals appear in this lemma can be understood in analogy to the method of *integration along characteristics* (see e.g. [**Ta, Fl**]) for a solution of an inhomogeneous wave equation (for a more detailed discussion of this point see [**F5**]). The advantage of the above lemma is that it gives a whole series of line integrals. A further difference is that the left side is an operator product, making it unnecessary to specify initial or boundary values.

Proof of Lemma 2.5.2. Our method is to first compute the Laplacian of both sides of (2.5.22). Comparing the structure of the resulting formulas, it will be possible to proceed by induction in l.

On the left side of (2.5.22), we calculate the Laplacian with the help of (2.5.15) to

$$-\Box_x (S^{(l)}\, V\, S^{(r)})(x,y) \;=\; \delta_{l,0}\, V(x)\, S^{(r)}(x,y) \;+\; l\,(S^{(l-1)}\, V\, S^{(r)})(x,y)\,. \tag{2.5.23}$$

The Laplacian of the integral on the right side of (2.5.22) can be computed with (2.5.21) and (2.5.15),

$$\begin{aligned}
&-\Box_x \int_0^1 \alpha^l\,(1-\alpha)^r\,(\alpha-\alpha^2)^n\,(\Box^n V)_{|\alpha y+(1-\alpha)x}\,d\alpha\; S^{(n+l+r+1)}(x,y) \qquad (2.5.24)\\
&= -\int_0^1 \alpha^l\,(1-\alpha)^{r+2}\,(\alpha-\alpha^2)^n\,(\Box^{n+1} V)_{|\alpha y+(1-\alpha)x}\,d\alpha\; S^{(n+l+r+1)}(x,y)\\
&\quad -\int_0^1 \alpha^l\,(1-\alpha)^{r+1}\,(\alpha-\alpha^2)^n\\
&\qquad\qquad \times (\partial_k \Box^n V)_{|\alpha y+(1-\alpha)x}\,d\alpha\,(y-x)^k\; S^{(n+l+r)}(x,y)\\
&\quad +(n+l+r+1)\\
&\qquad \times \int_0^1 \alpha^l\,(1-\alpha)^r\,(\alpha-\alpha^2)^n (\Box^n V)_{|\alpha y+(1-\alpha)x}\,d\alpha\; S^{(n+l+r)}(x,y)\,.
\end{aligned}$$

In the second summand, we rewrite the partial derivative as a derivative with respect to α and integrate by parts,

$$\begin{aligned}
&\int_0^1 \alpha^l\,(1-\alpha)^{r+1}\,(\alpha-\alpha^2)^n\,(\partial_k \Box^n V)_{|\alpha y+(1-\alpha)x}\,d\alpha\,(y-x)^k\\
&= \int_0^1 \alpha^l\,(1-\alpha)^{r+1}\,(\alpha-\alpha^2)^n\,\frac{d}{d\alpha}(\Box^n V)_{|\alpha y+(1-\alpha)x}\,d\alpha\\
&= -\delta_{n,0}\,\delta_{l,0}\,V(x)\\
&\quad -(n+l)\int_0^1 \alpha^l\,(1-\alpha)^{r+2}\,(\alpha-\alpha^2)^{n-1}\,(\Box^n V)_{|\alpha y+(1-\alpha)x}\,d\alpha\\
&\quad +(n+r+1)\int_0^1 \alpha^l\,(1-\alpha)^r\,(\alpha-\alpha^2)^n\,(\Box^n V)_{|\alpha y+(1-\alpha)x}\,d\alpha\\
&= -\delta_{n,0}\,\delta_{l,0}\,V(x)\\
&\quad -n\int_0^1 \alpha^l\,(1-\alpha)^{r+2}\,(\alpha-\alpha^2)^{n-1}\,(\Box^n V)_{|\alpha y+(1-\alpha)x}\,d\alpha\\
&\quad +(n+l+r+1)\int_0^1 \alpha^l\,(1-\alpha)^r\,(\alpha-\alpha^2)^n\,(\Box^n V)_{|\alpha y+(1-\alpha)x}\,d\alpha\\
&\quad -l\int_0^1 \alpha^{l-1}\,(1-\alpha)^r\,(\alpha-\alpha^2)^n\,(\Box^n V)_{|\alpha y+(1-\alpha)x}\,d\alpha\,.
\end{aligned}$$

We substitute back into the original equation to obtain

$$\begin{aligned}
(2.5.24) \;&=\; \delta_{n,0}\,\delta_{l,0}\,V(x)\,S^{(r)}(x,y)\\
&\quad +l\int_0^1 \alpha^{l-1}\,(1-\alpha)^r\,(\alpha-\alpha^2)^n\,(\Box^n V)_{|\alpha y+(1-\alpha)x}\,d\alpha\; S^{(n+l+r)}(x,y)\\
&\quad -\int_0^1 \alpha^l\,(1-\alpha)^{r+2}\,(\alpha-\alpha^2)^n\,(\Box^{n+1} V)_{|\alpha y+(1-\alpha)x}\,d\alpha\; S^{(n+l+r+1)}(x,y)\\
&\quad +n\int_0^1 \alpha^l\,(1-\alpha)^{r+2}\,(\alpha-\alpha^2)^{n-1}\,(\Box^n V)_{|\alpha y+(1-\alpha)x}\,d\alpha\; S^{(n+l+r)}(x,y)\,.
\end{aligned}$$

After dividing by $n!$ and summing over n, the last two summands are telescopic and cancel each other. Thus we get

$$\begin{aligned} -\Box \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^1 \alpha^l \, (1-\alpha)^r \, (\alpha-\alpha^2)^n \, (\Box^n V)_{|\alpha y+(1-\alpha)x} \, d\alpha \; S^{(n+l+r+1)}(x,y) \\ = \;\; \delta_{l,0} \, V(x) \, S^{(r)}(x,y) \;+\; l \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^1 \alpha^{l-1} \, (1-\alpha)^r \, (\alpha-\alpha^2)^n \\ \times \, (\Box^n V)_{|\alpha y+(1-\alpha)x} \, d\alpha \; S^{(n+l+r)}(x,y) \,. \end{aligned} \tag{2.5.25}$$

We now compare the formulas (2.5.23) and (2.5.25) for the Laplacian of both sides of (2.5.22). In the special case $l = 0$, these formulas coincide, and we can use a uniqueness argument for the solutions of the wave equation to prove (2.5.22): We assume that we consider the advanced Green's function (for the retarded Green's function, the argument is analogous). For given y, we denote the difference of both sides of (2.5.22) by $F(x)$. Since the support of $F(x)$ is in the past light cone $x \in L_y^\wedge$, F vanishes in a neighborhood of the hypersurface $\mathcal{H} = \{z \in \mathbb{R}^4 \,|\, z^0 = y^0 + 1\}$. Moreover, the Laplacian of F is identically equal to zero according to (2.5.23) and (2.5.25). We conclude that

$$\Box F \;=\; 0 \qquad \text{and} \qquad F_{|\mathcal{H}} \;=\; \partial_k F_{|\mathcal{H}} \;=\; 0 \,.$$

Since the wave equation has a unique solution for given initial data on the Cauchy surface $\mathcal{H}$, F vanishes identically.

The general case follows by induction in l: Suppose that (2.5.22) holds for given $\hat{l}$ (and arbitrary r). Then, according to (2.5.23, 2.5.25) and the induction hypothesis, the Laplacian of both sides of (2.5.22) coincides for $l = \hat{l} + 1$. The above uniqueness argument for the solutions of the wave equation again gives (2.5.22). ∎

The above lemma can be used iteratively for the light-cone expansion of more complicated operator products. To explain the method, we look at the example of three factors $S^{(0)}$ and two potentials V, W,

$$(S^{(0)} \, V \, S^{(0)} \, W \, S^{(0)})(x,y) \;=\; \int d^4z \, S^{(0)}(x,z) \, V(z) \, (S^{(0)} \, W \, S^{(0)})(z,y) \,. \tag{2.5.26}$$

Having split up the operator product in this form, we can apply Lemma 2.5.2 to the factor $S^{(0)} W S^{(0)}$,

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \int d^4z \, S^{(0)}(x,z) \left\{ V(z) \int_0^1 (\alpha-\alpha^2)^n \, (\Box^n W)_{|\alpha y+(1-\alpha)z} \, d\alpha \right\} S^{(n+1)}(z,y) \,.$$

Now we rewrite the z-integral as the operator product $(S^{(0)} g_y S^{(0)})(x,y)$, where $g_y(z)$ is the function in the curly brackets. The y-dependence of g_y causes no problems because we can view y as a fixed parameter throughout. Thus we can simply apply Lemma 2.5.2 once again to obtain

$$\begin{aligned} = \;\; \sum_{m,n=0}^{\infty} \frac{1}{m!\, n!} \int_0^1 d\beta \, (1-\beta)^{n+1} \, (\beta-\beta^2)^m \int_0^1 d\alpha \, (\alpha-\alpha^2)^n \\ \times \, \Box_z^m \left(V(z) \, (\Box^n W)_{|\alpha y+(1-\alpha)z} \right)_{|z=\beta y+(1-\beta)x} \; S^{(m+n+2)}(x,y) \,. \end{aligned}$$

Now the Laplacian $\Box_z^m$ can be carried out with the Leibniz rule. Notice that the manipulations of the infinite series are not problematic because the number of terms is finite to every order on the light cone.

The Feynman diagrams in (2.5.13) can be expanded with this iterative method, but they are a bit more difficult to handle. One complication is that pulling the Dirac matrices out of the Green's functions according to (2.5.12) gives one additional partial derivative per Green's function. However, this causes no major problems, because these partial derivatives can be carried out after each induction step by differentiating through the light-cone expansion of Lemma 2.5.2. Another issue is to keep track of the *chirality* of the potentials. To this end, one must use that the zero mass Green's function s and the factors $\not{A}_{L/R}$ are odd, whereas the dynamical mass matrices are even. Thus the chirality of the potentials changes each time a dynamical mass matrix appears, as in the example of the operator product

$$\chi_L \, s \, \not{A}_L \, s \cdots s \, \not{A}_L \, s \, Y_L \, s \, \not{A}_R \, s \cdots s \, \not{A}_R \, s \, Y_R \, s \, \not{A}_L \, s \cdots . \tag{2.5.27}$$

The last difficulty is that partial derivatives inside the line integrals may be contracted with a factor $(y-x)$, like for example in the expression

$$\int_0^1 (y-x)^j \, \partial_j V_{|\alpha y + (1-\alpha) x} \, d\alpha \,.$$

Such derivatives act in direction of the line integral and are thus called *tangential*. Writing them as derivatives with respect to the integration variable, we can integrate by parts, e.g.

$$\int_0^1 (y-x)^j \, \partial_j V_{|\alpha y + (1-\alpha) x} \, d\alpha \;=\; \int_0^1 \frac{d}{d\alpha} V(\alpha y + (1-\alpha) x) \, d\alpha \;=\; V(y) - V(x) \,.$$

Going through the calculations and the combinatorics in detail, one finds that with such integrations by parts we can indeed get rid of all tangential derivatives, and one ends up with terms of th following structure (for the proof see [**F6**]).

THEOREM 2.5.3. **(light-cone expansion of the k^{th} order Feynman diagram)** *Using a multi-index notation and the abbreviation*

$$\int_x^y [l, r \,|\, n] \, f(z) \, dz \;:=\; \int_0^1 d\alpha \; \alpha^l \, (1-\alpha)^r \, (\alpha - \alpha^2)^n \; f(\alpha y + (1-\alpha) x) \,, \tag{2.5.28}$$

the light-cone expansion of the k^{th} order contribution to the perturbation series (2.5.13) can be written as an infinite sum of expressions of the form

$$\begin{aligned} \chi_c \, C \, (y-x)^K \, W^{(0)}(x) \int_x^y [l_1, r_1 \,|\, n_1] \, dz_1 \; W^{(1)}(z_1) \int_{z_1}^y [l_2, r_2 \,|\, n_2] \, dz_2 \; W^{(2)}(z_2) \\ \cdots \int_{z_{\alpha-1}}^y [l_\alpha, r_\alpha \,|\, n_\alpha] \, dz_\alpha \; W^{(\alpha)}(z_\alpha) \; \gamma^J \; S^{(h)}(x,y) \quad , \; \alpha \leq k \,. \end{aligned} \tag{2.5.29}$$

Here the factors $W^{(\beta)}$ are composed of the potentials and their partial derivatives,

$$W^{(\beta)} \;=\; (\partial^{K_{a_\beta}} \Box^{p_{a_\beta}} V^{(a_\beta)}_{J_{a_\beta}, c_{a_\beta}}) \cdots (\partial^{K_{b_\beta}} \Box^{p_{b_\beta}} V^{(b_\beta)}_{J_{b_\beta}, c_{b_\beta}}) \tag{2.5.30}$$

with $a_1 = 1$, $a_{\beta+1} = b_\beta + 1$, $b_\beta \geq a_\beta - 1$ (in the case $b_\beta = a_\beta - 1$, $W^{(\beta)}$ is identically equal to one) and $b_\alpha = k$. Furthermore, $c, c_a \in \{L, R\}$ are chiral indices, C is a complex number, and the parameters l_a, r_a, n_a, and p_a are non-negative integers. The

functions $V^{(a)}_{J_a,c_a}$ coincide with any of the individual potentials in (2.5.11) with chirality c_a, i.e.

$$\begin{aligned} V^{(a)}_{c_a} &= A_{c_a} \qquad \textit{(in which case } |J_a|=1\textit{)} \quad \textit{or} \\ V^{(a)}_{c_a} &= m Y_{c_a} \qquad \textit{(in which case } |J_a|=0\textit{)}\,. \end{aligned} \tag{2.5.31}$$

The chirality c_a of the potentials is determined by the following rules:

(i) The chirality c_1 of the first potential coincides with the chirality of the factor χ_c.

(ii) The chirality of the potentials is reversed at every mass matrix, i.e.

$$c_a \text{ and } c_{a+1} \quad \begin{cases} \textit{coincide} & \textit{if } V^{(a)}_{c_a} = A_{c_a} \\ \textit{are opposite} & \textit{if } V^{(a)}_{c_a} = m Y_{c_a}\,. \end{cases}$$

The tensor indices of the multi-indices are all contracted with each other, according to the following rules:

(a) No two tensor indices of the same multi-index are contracted with each other.

(b) The tensor indices of the factor γ^J are all contracted with different multi-indices.

(c) The tensor indices of the factor $(y-x)^K$ are all contracted with the tensor indices of the factors $V^{(a)}_{J_a}$ or γ^J, but not with the partial derivatives ∂^{K_a}.

To every order h on the light cone, the number of terms of the form (2.5.29) is finite. Furthermore,

$$2h \;=\; k - 1 - |K| + \sum_{a=1}^{k} (|K_a| + 2p_a)\,. \tag{2.5.32}$$

The rules *(i)* and *(ii)* correspond precisely to our observation that the chirality changes at each dynamical mass matrix (2.5.27). The restrictions *(a)* and *(b)* on the possible contractions of tensor indices prevent an abuse of our multi-index notation. More precisely, *(a)* avoids factors $(y-x)^2$ in $(y-x)^I$, an unnecessary large number of γ-matrices in γ^J and "hidden" Laplacians inside the partial derivatives $\partial^{I_a}_{z_a}$. The rule *(b)*, on the other hand, prevents factors $(y-x)^2$ and hidden Laplacians in combinations of the form $(y-x)_i\,(y-x)_j\,\gamma^i\,\gamma^j$ and $\partial_{ij} V^{(a)}_{J_a}\,\gamma^i\,\gamma^j$, respectively. The rule *(c)* means that no tangential derivatives appear. The rules *(a)–(c)* imply that the tensor indices of the multi-index K are all contracted with the chiral potentials, except for one index which may be contracted with the factor γ^J. Since at most k chiral potentials appear, we get the inequality $|K| \leq k+1$. Using this inequality in (2.5.32) we get the bound

$$h \;\geq\; -1 + \frac{1}{2} \sum_{a=1}^{k} (|K_a| + 2p_a)\,. \tag{2.5.33}$$

This shows that h never becomes smaller than -1 and that derivatives of the potentials increase the order on the light cone. In the case $h = -1$, it follows from (2.5.32) that $|K| \geq 1$, meaning that at least one factor $(y-x)$ appears in (2.5.29). We conclude that the factor $S^{(h)}$ in (2.5.29) is always well-defined by either (2.5.14) or (2.5.21).

So far the Green's function was described perturbatively by a sum of Feynman diagrams (2.5.13). In order to get from this perturbative description to *non-perturbative* formulas of the light-cone expansion, we shall now carry out the sum over all Feynman diagrams to any fixed order on the light cone. This procedure is called *resummation* of

the light-cone expansion. In order to give a first idea of how the resummation works, we consider the leading singularity on the light cone by neglecting all terms of the order $\mathcal{O}((y-x)^{-2})$. According to (2.5.7), we need to take into account only the contributions (2.5.29) with $h = -1$. The inequality (2.5.33) implies that no derivatives of the potentials appear. Moreover, we obtain from (2.5.32) that $|K| = k+1$. Again using the contraction rules *(a)–(c)*, we conclude that one tensor index of the multi-index K is contracted with a Dirac matrix, whereas the remaining k indices of K are all contracted with chiral potentials. Therefore, all k potentials are chiral, and no dynamical mass matrices appear. A detailed calculation yields for the k^{th} order Feynman diagram a term of precisely this structure,

$$\begin{aligned}\chi_c\,((-s\,B)^k s)(x,y) \;&=\; \chi_c\,(-i)^k \int_x^y dz_1\;(y-x)_{j_1}\,A_c^{j_1}(z_1) \\ &\times \int_{z_1}^y dz_2\;(y-z_1)_{j_2}\,A_c^{j_2}(z_2) \cdots \int_{z_{k-1}}^y dz_k\;(y-z_k)_{j_2}\,A_c^{j_k}(z_k)\;s(x,y) \\ &+ \mathcal{O}((y-x)^{-2})\,,\end{aligned}$$

where we used for the line integrals the short notation

$$\int_x^y f(z)\,dz \;:=\; \int_x^y [0,0\,|\,0]\,f(z)\,dz \;=\; \int_0^1 f(\alpha y + (1-\alpha)x)\,d\alpha\,. \tag{2.5.34}$$

The obtained nested line integrals can be identified with the summands of the familiar Dyson series. This allows us to carry out the sum over all Feynman diagrams,

$$\chi_c\,\tilde{s}(x,y) \;=\; \chi_c\,\text{Pexp}\left(-i\int_x^y (y-x)_j\,A_c^j(z)\,dz\right) s(x,y) \;+\; \mathcal{O}((y-x)^{-2})\,, \tag{2.5.35}$$

where we again used the notation (2.5.34) and the following definition of Pexp.

DEF. 2.5.4. *For a smooth one-parameter family of matrices $F(\alpha)$, $\alpha \in \mathbb{R}$, the* **ordered exponential** $\text{Pexp}(\int F(\alpha)\,d\alpha)$ *is given by the Dyson series*

$$\begin{aligned}\text{Pexp}\left(\int_a^b F(\alpha)\,d\alpha\right) \;&=\; \mathbb{1} \;+\; \int_a^b dt_0\,F(t_0)\,dt_0 \;+\; \int_a^b dt_0\,F(t_0)\int_{t_0}^b dt_1\,F(t_1) \\ &+ \int_a^b dt_0\,F(t_0)\int_{t_0}^b dt_1\,F(t_1)\int_{t_1}^b dt_2\,F(t_2) \;+\; \cdots\,.\end{aligned} \tag{2.5.36}$$

The appearance of the ordered exponential in (2.5.35) can be understood from the local gauge invariance. We explain this relation for simplicity in the example of dynamical mass matrices and chiral potentials of the form

$$Y_L \;=\; Y_R \;=\; 0\,, \qquad A_L(x) \;=\; A_R(x) \;=\; iU(x)\,(\partial U^{-1}(x))\,,$$

where $U = U^{(a\alpha)}_{(b\beta)}$ is a unitary matrix on the sectors and generations. In this case, the Dirac operator is related to the free Dirac operator simply by a local unitary transformation,

$$i\partial\!\!\!/ + B \;=\; U\,i\partial\!\!\!/\,U^{-1}\,.$$

Interpreting this local transformation as in §1.4 as a gauge transformation (1.4.3), we can say that the external potential can be transformed away globally by choosing a suitable gauge. Using the well-known behavior of the Green's function under gauge transformations, we obtain the simple formula

$$\tilde{s}(x,y) \;=\; U(x)\,s(x,y)\,U^{-1}(y)\,. \tag{2.5.37}$$

Let us verify that this is consistent with (2.5.35). Setting $V(\alpha) = U(\alpha y + (1-\alpha)x)$ and using the relation $V(V^{-1})' = (VV^{-1})' - V'V^{-1} = -V'V^{-1}$, we can write the integrand of the ordered exponential as

$$-i(y-x)_j\, A_c^j(z) \;=\; -V'(\alpha)\, V^{-1}(\alpha)\,. \tag{2.5.38}$$

Differentiating (2.5.36) with respect to a as well as evaluating it for $a = b$, one sees that the ordered exponential can be characterized as the solution of the initial value problem

$$\frac{d}{da}\text{Pexp}\left(\int_a^b F\right) \;=\; -F(a)\,\text{Pexp}\left(\int_a^b F\right), \qquad \text{Pexp}\left(\int_b^b F\right) \;=\; \mathbb{1}\,.$$

In the case $F = -V'V^{-1}$, it is easily verified that the solution to this initial value problem is

$$\text{Pexp}\left(-\int_a^b V'(\alpha)\, V^{-1}(\alpha)\, d\alpha\right) \;=\; V(a)\, V^{-1}(b)\,.$$

Using (2.5.38), we conclude that

$$\text{Pexp}\left(-i\int_x^y (y-x)_j\, A_c^j(z)\, dz\right) \;=\; U(x)\, U^{-1}(y)\,.$$

Thus the ordered exponential in (2.5.35) gives precisely the factor $U(x)\, U(y)^{-1}$ needed for the correct behavior under gauge transformations (2.5.37).

To higher order on the light cone, the situation clearly is more complicated. Nevertheless, it is very helpful to imagine that after rearranging the summands of the light-cone expansion in a suitable way, certain subseries can be summed up explicitly giving rise to ordered exponentials of the chiral potentials. As in (2.5.27), the chirality of the potentials should change each time a dynamical mass matrix appears. This conception is made precise by the following definition and theorem, proving that the light-cone expansion of the Green's function can be obtained to any given order on the light cone by taking a finite number of terms of the form (2.5.28) and inserting suitable ordered exponentials.

DEF. 2.5.5. *A contribution (2.5.28) to the light-cone expansion of Theorem 2.5.3 is called* **phase-free** *if all the tangential potentials $V_{J_a}^{(a)}$ are differentiated, i.e.*

$$|K_a| + 2p_a > 0 \qquad \textit{whenever} \qquad J_a \textit{ is contracted with } (y-x)^K.$$

From every phase-free contribution the corresponding **phase-inserted** *contribution is obtained as follows: We insert ordered exponentials according to the replacement rule*

$$W^{(\beta)}(z_\beta) \;\longrightarrow\; W^{(\beta)}(z_\beta)\,\text{Pexp}\left(-i\int_{z_\beta}^{z_{\beta+1}} A_{c_\beta}^{j_\beta}\,(z_{\beta+1}-z_\beta)\right), \quad \beta = 0,\ldots,\alpha\,,$$

where we set $z_0 = x$ and $z_{\alpha+1} = y$. The chiralities c_β are determined by the relations $c_0 = c$ and

$$c_{\beta-1} \textit{ and } c_\beta \left\{\begin{matrix} \textit{coincide} \\ \textit{are opposite}\end{matrix}\right\} \textit{ if } W^{(\beta-1)} \textit{ contains an } \left\{\begin{matrix}\textit{even}\\ \textit{odd}\end{matrix}\right\} \textit{number of factors } Y_\cdot.$$

THEOREM 2.5.6. *To every order on the light cone, the number of phase-free contributions is finite. The light-cone expansion of the Green's function $\tilde{s}(x,y)$ is given by the sum of the corresponding phase-inserted contributions.*

For the proof we refer to [**F6**]. In short, the first statement follows directly from the contraction rules *(a)–(c)* and (2.5.32), whereas for the second part one uses for fixed x and y the behavior of the Green's function under non-unitary local transformations of the spinors.

Our constructions have led to a convenient procedure for performing the light-cone expansion of the Green's function. One only needs to compute to any order on the light cone the finite number of phase-free contributions. Then one inserts ordered exponentials according to Def. 2.5.5. For the computation of the phase-free contributions we use a computer program. Appendix B lists those phase-free contributions which will be of relevance in Chapters 6–8.

In the remainder of this section we describe how the above methods can be adapted to the *fermionic projector*. We begin for simplicity with the fermionic projector corresponding to one Dirac sea in the vacuum (2.2.1). Similar to (2.5.4) we pull out the Dirac matrices,

$$P^{\text{sea}}(x,y) \;=\; (i\partial\!\!\!/_x + m)\, T_{m^2}(x,y)\,, \tag{2.5.39}$$

where T_{m^2} is the Fourier transform of the lower mass shell,

$$T_{m^2}(x,y) \;=\; \int \frac{d^4k}{(2\pi)^4}\, \delta(k^2-m^2)\, \Theta(-k^0)\, e^{-ik(x-y)}\,. \tag{2.5.40}$$

Computing this Fourier integral and expanding the resulting Bessel functions gives

$$\begin{aligned} T_{m^2}(x,y) \;=\; & -\frac{1}{8\pi^3}\left(\frac{\text{PP}}{\xi^2} \;+\; i\pi\delta(\xi^2)\,\varepsilon(\xi^0)\right) \\ & + \frac{m^2}{32\pi^3}\left(\log|m^2\xi^2| + C_E + i\pi\,\Theta(\xi^2)\,\epsilon(\xi^0)\right) \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,(j+1)!}\,\frac{(m^2\xi^2)^j}{4^j} \\ & - \frac{m^2}{32\pi^3} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,(j+1)!}\,\frac{(m^2\xi^2)^j}{4^j}\,(\Phi(j+1)+\Phi(j))\,. \end{aligned} \tag{2.5.41}$$

Here $\xi \equiv y-x$, $C_E = 2C - 2\log 2$ with Euler's constant C, and Φ is the function

$$\Phi(0) = 0\,, \qquad \Phi(n) = \sum_{k=1}^{n} \frac{1}{k} \quad \text{for } n \geq 1\,.$$

Similar to (2.5.6), this expansion involves distributions which are singular on the light cone. But in addition to singularities $\sim \delta((y-x)^2$ and $\sim \Theta((y-x)^2$, now also singularities of the form form $\text{PP}/(y-x)^2$ and $\log|(y-x)^2|$ appear. In particular, T_{m^2} is *not causal* in the sense that $\operatorname{supp} T_{m^2}(x,.) \not\subset L_x$. Another similarity to (2.5.6) is that power series in $m^2(y-x)^2$ appear. This suggests that in analogy to (2.5.7), the higher orders in m^2 should be of higher order on the light cone. However, due to the term $\log|m^2\xi^2|$, the distribution T_{m^2} is *not* a power series in m^2. This means that the higher mass derivatives of T_{m^2} do not exist, and the analog of (2.5.7) breaks down. This so-called *logarithmic mass problem* reflects a basic infrared problem in the light-cone expansion of the fermionic projector. In the vacuum, it can be resolved with the following simple construction. We subtract the problematic $\log|m^2|$-terms by setting

$$T^{\text{reg}}_{m^2}(x,y) \;=\; T_{m^2}(x,y) \;-\; \frac{m^2}{32\pi^3}\,\log|m^2| \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,(j+1)!}\,\frac{(m^2\xi^2)^j}{4^j}\,. \tag{2.5.42}$$

This distribution is a power series in m^2, and in analogy to (2.5.14) we can set

$$T^{(l)} \;=\; \left(\frac{d}{da}\right)^l T^{\text{reg}}_a|_{a=0} \qquad\qquad (l \geq 0). \tag{2.5.43}$$

Furthermore, we introduce $T^{(-1)}$ similar to (2.5.21) as the distributional derivative of $T^{(0)}$. Similar to (2.5.7),

$$T^{(n)}(x,y) \qquad \text{is of the order} \quad \mathcal{O}((y-x)^{2n-2})\,,$$

and thus the mass expansion of $T^{\text{reg}}_{m^2}$ gives us its light-cone expansion. The point is that the difference of T_{m^2} and $T^{\text{reg}}_{m^2}$,

$$T_{m^2} - T^{\text{reg}}_{m^2} \;=\; \frac{m^2}{32\pi^3}\, \log|m^2| \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,(j+1)!}\, \frac{(m^2\xi^2)^j}{4^j}\,,$$

is an absolutely convergent power series in ξ^2 and is thus a *smooth* function in position space. This smooth contribution is of no relevance as long as the singularities on the light cone are concerned. This leads us to write the fermionic projector in the form

$$P^{\text{sea}}(x,y) \;=\; \sum_{n=0}^{\infty} \frac{m^{2n}}{n!} (i\partial\!\!\!/_x + m)\, T^{(n)}(x,y) \;+\; P^{\text{lc}}(x,y)\,, \tag{2.5.44}$$

with

$$P^{\text{lc}} \;:=\; (i\partial\!\!\!/_x + m)\left(T_{m^2} - T^{\text{reg}}_{m^2}\right).$$

The series in (2.5.44) is a light-cone expansion which completely describes the singular behavior of the fermionic projector on the light cone. The so-called *low-energy contribution* P^{lc}, on the other hand, is a smooth function.

This method of performing a light-cone expansion modulo smooth functions on the light cone also works for the general fermionic projector with interaction. But the situation is more complicated and at the same time more interesting, in particular because the space-time dependence of the involved external potentials reveals the causal structure of the fermionic projector. The first construction step is to use the identity on the left side of (2.3.12) to carry over the light-cone expansion from the Green's function to the distribution $\tilde{k}$. Next, comparing (2.2.6) with the formula

$$\begin{aligned} p_m(k) &= (k\!\!\!/ + m)\, \delta(k^2 - m^2) \\ &= \frac{1}{2\pi i}\, (k\!\!\!/ + m) \lim_{\varepsilon \searrow 0} \left[\frac{1}{k^2 - m^2 - i\varepsilon} \;-\; \frac{1}{k^2 - m^2 + i\varepsilon} \right], \end{aligned}$$

one sees that the distributions p and k differ from each only by the $i\varepsilon$-regularization in momentum space. The key step of the construction is the so-called *residual argument*, which relates the light-cone expansion of $\tilde{k}$ to an expansion in momentum space. Using that the latter expansion remains unchanged if the poles of the distributions are suitably shifted in momentum space, one obtains the light-cone expansion for an operator $\tilde{p}^{\text{res}}$, which can be regarded as a perturbation of p, but with a different combinatorics than in the expansion of Theorem 2.3.1. More precisely, $\tilde{p}$ can be obtained from $\tilde{p}^{\text{res}}$ by replacing pairs of factors k in the perturbation expansion by corresponding factors p. The argument (2.4.2) shows that the difference $\tilde{p} - \tilde{p}^{\text{res}}$ vanishes in the static case, and more generally one can say that $\tilde{p} - \tilde{p}^{\text{res}}$ will be of

significance only if the frequency of the external potentials is of the order of the mass gap. Therefore, the operator

$$P^{\text{he}} := \frac{1}{2} X (\tilde{p} - \tilde{p}^{\text{res}})$$

is called the *high-energy contribution* to the fermionic projector. Moreover, resolving the logarithmic mass problem we obtain again a *low-energy contribution* P^{le}. We thus obain a representation of the fermionic projector of the form

$$\begin{aligned} P^{\text{sea}}(x,y) &= \sum_{n=-1}^{\infty} (\text{phase-inserted line integrals}) \times T^{(n)}(x,y) \\ &\quad + P^{\text{le}}(x,y) + P^{\text{he}}(x,y) \,. \end{aligned} \tag{2.5.45}$$

Here the series is a light-cone expansion which describes the singular behavior of the fermionic projector on the light cone non-perturbatively. It is obtained from the light-cone expansion of the Green's functions by the simple replacement rule

$$S^{(n)} \longrightarrow T^{(n)} \,.$$

In particular, the phase-inserted line integrals are exactly the same as those for the Green's functions (see Def. 2.5.5). The contributions P^{le} and P^{he}, on the other hand, are given perturbatively by a series of terms which are all smooth on the light cone (we expect that the perturbation series for P^{le} and P^{le} should converge, but this has not yet been proven). The "causality" of the causal perturbation expansion can be understood from the fact that the phase-inserted line integrals in (2.5.45) are all bounded integrals along the line segment joining the points x and y (whereas the light-cone expansion of general operator products involves unbounded line integrals [**F1**]). In particular, when y lies in the causal future or past of x, the light-cone expansion in (2.5.45) depends on the external potential only inside the "diamond" $(J_x^\vee \cap J_y^\wedge) \cup (J_x^\wedge \cap J_y^\vee)$. Nevertheless, the light-cone expansion is not causal in the strict sense because there are contributions for $y \notin J_x$. Furthermore, the low- and high-energy contributions cannot be described with line integrals and violate locality as well as causality. This non-locality can be understood from the fact that the fermionic projector is a global object in space-time (see the discussion in §2.4). Mathematically, it is a consequence of the non-local operation of taking the absolute value of an operator (2.2.17) in the definition of the fermionic projector. We conclude that the singular behavior of the fermionic projector on the light-cone can be described explicitly by causal line integrals, whereas the smooth contributions to the fermionic projector are governed by non-local effects.

Inspecting the explicit formulas of Appendix B, one sees immediately that from the line integrals of the light-cone expansion one can reconstruct the chiral and scalar/pseudo-scalar potentials. In this sense, P^{sea} encodes all information on the external potential. Furthermore, the fermionic projector gives via its representation (2.3.19) all information on the fermions and anti-fermions of the system. We thus come to the important conclusion that the fermionic projector describes the physical system completely.

2.6. Normalization of the Fermionic States

In §2.2 we disregarded that the fermionic states are in general not normalizable in infinite volume. We avoided this problem using a δ-normalization in the mass parameter (see e.g. (2.2.26)). In this section we will analyze the normalization in

detail by considering the system in finite volume and taking the infinite volume limit. Apart from justifying the formalism introduced in §2.2, this will clarify in which sense the fermionic projector is idempotent (Theorem 2.6.1). Furthermore, we will see that the probability integral has a well-defined infinite volume limit (Theorem 2.6.2), and this will also determine the normalization constant c_{norm} in (2.3.19) (see (2.6.24)). We postpone the complications related to the chiral fermions to Appendix C and thus assume here that the chiral asymmetry matrix $X = \mathbb{1}$. We work again in the setting of §2.3 and assume that the external potential $\mathcal{B}$ satisfies the regularity assumptions of Lemma 2.2.2. Furthermore, we make the physically reasonable assumption that the *masses* are *non-degenerate in the generations*, meaning that

$$m_{a\alpha} \neq m_{a\beta} \qquad \text{for all } a \text{ and } \alpha \neq \beta\,. \tag{2.6.1}$$

In order to ensure that all normalization integrals are finite, we need to introduce an *infrared regularization*. For clarity, we explain the construction for a single Dirac sea of mass m in the vacuum. First, we replace space by the three-dimensional box

$$T^3 \;=\; [-l_1, l_1] \times [-l_2, l_2] \times [-l_3, l_3] \qquad \text{with } 0 < l_i < \infty \tag{2.6.2}$$

and set $V = |T^3| = 8\, l_1 l_2 l_3$. We impose periodic boundary conditions; this means that we restrict the momenta $\vec{k}$ to the lattice L^3 given by

$$L^3 \;=\; \frac{\pi\,\mathbb{Z}}{l_1} \times \frac{\pi\,\mathbb{Z}}{l_2} \times \frac{\pi\,\mathbb{Z}}{l_3} \;\subset\; \mathbb{R}^3\,.$$

In order to carry over the operators p_m, k_m and s_m (see §2.2) to finite volume, we leave the distributions in momentum space unchanged. In the transformation to position space, we replace the Fourier integral over 3-momentum by a Fourier series according to

$$\int \frac{d\vec{k}}{(2\pi)^3} \;\longrightarrow\; \frac{1}{V} \sum_{\vec{k} \in L^3}\,. \tag{2.6.3}$$

When taking products of the resulting operators, we must take into account that the spatial integral is now finite. For example, we obtain that

$$\begin{aligned}
(p_m\, p_{m'})(x,y) &= \int_{\mathbb{R}\times T^3} p_m(x,z)\, p_{m'}(z,y)\, d^4z \\
&= \int_{\mathbb{R}\times T^3} d^4z \int_{-\infty}^{\infty} \frac{dk^0}{2\pi}\, \frac{1}{V} \sum_{\vec{k}\in L^3} p_m(k)\, e^{-ik(x-z)} \\
&\qquad \times \int_{-\infty}^{\infty} \frac{dl^0}{2\pi}\, \frac{1}{V} \sum_{\vec{l}\in L^3} p_{m'}(l)\, e^{-il(z-y)} \\
&= \int_{-\infty}^{\infty} \frac{dk^0}{2\pi}\, \frac{1}{V} \sum_{\vec{k}\in L^3} p_m(k)\, e^{-ikx} \\
&\qquad \times \int_{-\infty}^{\infty} \frac{dl^0}{2\pi}\, \frac{1}{V} \sum_{\vec{l}\in L^3} p_{m'}(l)\, e^{ily}\, 2\pi\, \delta(k^0 - l^0)\, V\, \delta_{\vec{k},\vec{l}} \\
&= \int_{-\infty}^{\infty} \frac{dk^0}{2\pi}\, \frac{1}{V} \sum_{\vec{k}\in L^3} p_m(k)\, p_{m'}(k)\, e^{-ik(x-y)} \\
&= \delta(m-m')\, p_m(x,y)\,,
\end{aligned} \tag{2.6.4}$$

where $p_m(k) = (\not{k}+m)\,\delta(k^2-m^2)$. More generally, the calculation rules (2.2.26–2.2.32) for products of the operators k_m, p_m and s_m remain valid in finite 3-volume.

In (2.6.4) we are still using a δ-normalization in the mass parameter. In order to go beyond this formalism and to get into the position where we can multiply operators whose mass parameters coincide, we "average" the mass over a small interval $[m, m+\delta]$. More precisely, we set

$$\bar{p}_m \;=\; \frac{1}{\delta} \int_m^{m+\delta} p_\mu \, d\mu \qquad \text{and} \qquad \bar{k}_m \;=\; \frac{1}{\delta} \int_m^{m+\delta} k_\mu \, d\mu \;. \tag{2.6.5}$$

Then

$$\begin{aligned} \bar{p}_m \, \bar{p}_m &= \frac{1}{\delta^2} \int_m^{m+\delta} d\mu \int_m^{m+\delta} d\mu' \; p_\mu \, p_{\mu'} \\ &= \frac{1}{\delta^2} \int_m^{m+\delta} d\mu \int_m^{m+\delta} d\mu' \; \delta(\mu-\mu') \, p_\mu \;=\; \frac{1}{\delta^2} \int_m^{m+\delta} p_\mu \, d\mu \;=\; \frac{1}{\delta}\, \bar{p}_m \;, \end{aligned}$$

and thus, apart from the additional factor δ^{-1}, $\bar{p}_m$ is idempotent. Similarly, we have the relations

$$\bar{k}_m \, \bar{k}_m \;=\; \frac{1}{\delta}\, \bar{p}_m \qquad \text{and} \qquad \bar{k}_m \, \bar{p}_m \;=\; \bar{p}_m \, \bar{k}_m \;=\; \frac{1}{\delta}\, \bar{k}_m \;.$$

Thus, introducing the *infrared regularized fermionic projector* corresponding to a Dirac sea of mass m by

$$P^{\text{sea}} \;=\; \frac{\delta}{2} \,(\bar{p}_m - \bar{k}_m) \;, \tag{2.6.6}$$

this operator is indeed a projector,

$$(P^{\text{sea}})^* \;=\; P^{\text{sea}} \qquad \text{and} \qquad (P^{\text{sea}})^2 \;=\; P^{\text{sea}} \;. \tag{2.6.7}$$

The *infinite volume limit* corresponds to taking the limits $l_1, l_2, l_3 \to \infty$ and $\delta \searrow 0$.

Let us discuss the above construction. Clearly, our regularization method relies on special assumptions (3-dimensional box with periodic boundary conditions, averaging of the mass parameter). This is partly a matter of convenience, but partly also a necessity, because much more general regularizations would lead to unsurmountable technical difficulties. Generally speaking, infrared regularizations change the system only on the macroscopic scale near spatial infinity and possibly for large times. Due to the decay assumptions on the external potentials in Lemma 2.2.2, in this region the system is only weakly interacting. This should make infrared regularizations insensitive to the details of the regularization procedure, and it is reasonable to expect (although it is certainly not proven) that if the infinite volume limit exists, it should be independent of which regularization method is used. Here we simply take this assumption for granted and thus restrict attention to a special regularization scheme. But at least we will see that the infinite volume limit is independent of how the limits $l_i \to \infty$ and $\delta \searrow 0$ are taken.

It is worth noting that not *every* infrared regularization has a well-defined infinite volume limit. To see this, we consider the example of a regularization in a 4-dimensional box. Restricting the time integral to the finite interval $t \in [-T, T]$, we

obtain

$$\begin{aligned}
&(p_m\, p_m)(x,y) \\
&\quad = \int_{-T}^{T} dt \int_{-\infty}^{\infty} \frac{dk^0}{2\pi} \int_{-\infty}^{\infty} \frac{dl^0}{2\pi}\, \frac{1}{V} \sum_{\vec{k},\vec{l} \in L^3,\, \vec{k}=\vec{l}} e^{-i(k^0-l^0)t}\, p_m(k)\, p_m(l)\, e^{ikx-ily} \\
&\quad = \int_{-T}^{T} dt \int_{-\infty}^{\infty} \frac{dk^0}{2\pi} \int_{-\infty}^{\infty} \frac{dl^0}{2\pi}\, \frac{1}{V} \sum_{\vec{k},\vec{l} \in L^3,\, \vec{k}=\vec{l}} e^{-i(k^0-l^0)t} \\
&\qquad\qquad \times (\not{k} + m)\, (\not{l} + m)\, \delta((k^0)^2 - (l^0)^2)\, \delta(k^2 - m^2)\, e^{ikx-ily} \\
&\quad = \int_{-\infty}^{\infty} \frac{dk^0}{2\pi} \frac{1}{V} \sum_{\vec{k} \in L^3} \frac{mT}{|k^0|}\, p_m(k)\, e^{-ik(x-y)} \;+\; \mathcal{O}(T^0)\,,
\end{aligned}$$

and due to the factor $|k^0|^{-1}$ in the last line, this is not a multiple of $p_m(x,y)$. This problematic factor $|k^0|^{-1}$ also appears under more general circumstances (e.g. when we introduce boundary conditions at $t = \pm T$ and/or take averages of the mass parameter), and thus it seems impossible to arrange that the fermionic projector is idempotent. We conclude that a 4-dimensional box does not give a suitable regularization scheme.

The mass averaging in (2.6.5) leads to the bizarre effect that for fixed $\vec{k}$, a whole continuum of states of the fermionic projector, namely all states with

$$k^0 \in \left[-\sqrt{|\vec{k}|^2 + (m+\delta)^2},\; -\sqrt{|\vec{k}|^2 + m^2} \right], \tag{2.6.8}$$

are occupied. If one prefers to occupy only a finite number of states for every $\vec{k}$, one can achieve this by taking the mass averages for the bra- and ket-states separately. For example, instead of (2.6.6) we could define the fermionic projector by

$$P(x,y) \;=\; \delta \int_{-\infty}^{\infty} \frac{dk^0}{2\pi} \int_{-\infty}^{\infty} \frac{dl^0}{2\pi}\, \frac{1}{V} \sum_{\vec{k},\vec{l} \in L^3,\, \vec{k}=\vec{l}} \bar{t}_m(k)\, \bar{t}_m(l)\, e^{-ikx+ily} \tag{2.6.9}$$

with $\bar{t}_m = \frac{1}{2}(\bar{p}_m - \bar{k}_m)$. This fermionic projector is for every $\vec{k}$ composed of a finite number of states. Furthermore, it is a projector in the sense of (2.6.7). In contrast to (2.6.6), (2.6.9) is not homogeneous in time, but decays on the scale $t \sim \delta^{-1}$. However, if we restrict attention to a fixed region of space-time for which $t \ll \delta^{-1}$, then (2.6.6) and (2.6.9) differ only by terms of higher order in δ, and therefore we can expect that (2.6.5) and (2.6.7) should have the same infinite volume limit. The definition (2.6.6) has the advantage that it will be easier to introduce the interaction.

After these preparations, we come to the general construction of the fermionic projector in the three-dimensional box T^3. Since we want to "smear out" the mass similar to (2.6.5), the mass parameter needs to be variable. To this end, we introduce a parameter $\mu > 0$ which shifts all masses by the same amount. Thus we describe the system in the vacuum by the Dirac operator

$$i\partial\!\!\!/ - mY - \mu \mathbb{1}\,. \tag{2.6.10}$$

The external field is described by an operator $\mathcal{B}$ in the space-time $\mathbb{R} \times T^3$, which we again insert into the Dirac operator,

$$i\partial\!\!\!/ + \mathcal{B} - mY - \mu \mathbb{1}\,.$$

Now we can introduce the operators p, k and their perturbation expansions exactly as in §2.3. For clarity, we denote the dependence on the parameter μ by an additional index $+\mu$. In particular, we denote the operator products in Theorem 2.3.1 by $\tilde{p}_{+\mu}$ and $\tilde{k}_{+\mu}$. Since the multiplication rules (2.2.26–2.2.32) also hold in finite 3-volume, all the computations of §2.3 are still true. In particular, the operators $\tilde{p}_{+\mu}$ and $\tilde{k}_{+\mu}$ satisfy the δ-normalization conditions[3]

$$\tilde{p}_{+\mu}\, \tilde{p}_{+\mu'} \;=\; \tilde{k}_{+\mu}\, \tilde{k}_{+\mu'} \;=\; \delta(\mu-\mu')\, \tilde{p}_{+\mu} \tag{2.6.11}$$

$$\tilde{p}_{+\mu}\, \tilde{k}_{+\mu'} \;=\; \tilde{k}_{+\mu}\, \tilde{p}_{+\mu'} \;=\; \delta(\mu-\mu')\, \tilde{k}_{+\mu}\,. \tag{2.6.12}$$

In analogy to (2.6.5) and (2.6.6) we define the auxiliary fermionic projector by

$$P^{\text{sea}} \;=\; \frac{1}{2} \int_0^\delta (\tilde{p}_{+\mu} - \tilde{k}_{+\mu})\, d\mu\,, \tag{2.6.13}$$

and the fermionic projector is again obtained by taking the partial trace (2.3.20),

$$(P^{\text{sea}})^a_b \;=\; \sum_{\alpha=1}^{g(a)} \sum_{\beta=1}^{g(b)} (P^{\text{sea}})^{(a\alpha)}_{(b\beta)}\,. \tag{2.6.14}$$

The next theorem shows that the fermionic projector is idempotent in the infinite volume limit, independent of how the limits $l_i \to \infty$ and $\delta \searrow 0$ are taken.

THEOREM 2.6.1. **(idempotence of the fermionic projector)** *Consider a system composed of massive fermions with non-degenerate masses (2.6.1). Then the fermionic projector defined by (2.6.13) and (2.6.14) satisfies the relations*

$$\int_{\mathbb{R}\times T^3} d^4z \sum_{b=1}^{N} (P^{sea})^a_b(x,z)\, (P^{sea})^b_c(z,y) \;=\; (P^{sea})^a_c(x,y) \;+\; \delta^2\, Q^a_c(x,y)\,,$$

where Q has an expansion as a sum of operators which all have a well-defined infinite volume limit.

Proof. For simplicity we omit the superscript 'sea'. It follows immediately from (2.6.11–2.6.13) that the auxiliary fermionic projector is idempotent,

$$\sum_{b,\beta} P^{(a\alpha)}_{(b\beta)}\, P^{(b\beta)}_{(c\gamma)} \;=\; P^{(a\alpha)}_{(c\gamma)}\,. \tag{2.6.15}$$

Thus it remains to show that

$$\sum_b \sum_{\alpha,\gamma} \sum_{\beta,\beta' \text{ with } \beta\neq\beta'} P^{(a\alpha)}_{(b\beta)}\, P^{(b\beta)}_{(c\gamma)} \;=\; \delta^2\, Q^a_c(x,y)\,. \tag{2.6.16}$$

According to the non-degeneracy assumption (2.6.1), there are constants $c, \delta > 0$ such that for all sufficiently small δ,

$$|(m_{b\beta}+\mu) - (m_{b\beta'}+\mu')| \;\geq\; c \qquad \text{for all } b,\, \beta\neq\beta', \text{ and } 0<\mu,\mu'<\delta\,. \tag{2.6.17}$$

[3]Online version: As noticed by A. Grotz, these relations are in general violated to higher order in perturbation theory. In order to cure the problem, one needs to rescale the states of the fermionic projector, as is worked out in the paper arXiv:0901.0334 [math-ph].

On the left side of (2.6.16) we substitute (2.6.13) and the operator product expansions of Theorem 2.3.1. Using (2.6.17), the resulting operator products are all finite and can be estimated using the relations

$$\int_0^\delta d\mu \int_0^\delta d\mu' \,(\cdots A_{+\mu})^{(a\alpha)}_{(b\beta)} \,\big(A_{+\mu'} \cdots\big)^{(b\beta')}_{(c\gamma)} \;=\; c^{-1}\, \mathcal{O}(\delta^2)\,, \tag{2.6.18}$$

where each factor A stands for p, k or s. This gives (2.6.16). ∎

At this point we can make a remark on the name "partial trace." The notion of a trace suggests that two matrix indices should be set equal and summed over; thus one may want to define the fermionic projector instead of (2.3.20) by

$$P^a_b(x,y) \;=\; \sum_{\alpha=1}^3 P^{(a\alpha)}_{(b\alpha)}(x,y)\,. \tag{2.6.19}$$

This alternative definition suffers from the following problem. The off-diagonal elements of $P^{(a\alpha)}_{(b\beta)}$, $\alpha \neq \beta$, are important to make the auxiliary fermionic projector idempotent, because

$$P^{(a\alpha)}_{(c\gamma)} \overset{(2.6.15)}{=} \sum_{b=1}^8 \sum_{\beta=1}^3 P^{(a\alpha)}_{(b\beta)}\, P^{(b\beta)}_{(c\gamma)} \overset{\text{in general}}{\neq} \sum_{b=1}^8 P^{(a\alpha)}_{(b\alpha)}\, P^{(b\gamma)}_{(c\gamma)}\,.$$

But these off-diagonal elements do not enter the definition (2.6.19), and this makes it difficult to arrange that P^a_b is idempotent. In more technical terms, defining the fermionic projector by (2.6.19) would in the proof of the above theorem lead instead of (2.6.16) to the conditions

$$\sum_{b=1}^8 \sum_{\alpha,\beta=1}^3 \Big(P^{(a\alpha)}_{(b\alpha)}\, P^{(b\beta)}_{(c\beta)} \;-\; P^{(a\alpha)}_{(b\beta)}\, P^{(b\beta)}_{(c\alpha)}\Big) \;=\; \delta^2\, Q^a_c(x,y)\,.$$

As a consequence, we would in (2.6.18) get contributions with $\beta = \beta'$, which are singular. The only way to avoid these singular contributions would be to consider perturbations which are diagonal on the generations. But in this special case, also the auxiliary fermionic projector is diagonal on the generations, and so the definitions (2.6.19) and (2.3.20) coincide. We conclude that (2.3.20) is the more general and thus preferable definition of the partial trace.

We next consider the normalization of the individual states of the fermionic projector. In finite 3-volume in the vacuum, a Dirac sea of mass m is composed of a discrete number of fermionic states. More precisely,

$$\begin{aligned} P^{\text{sea}}(x,y) &= \int \frac{dk^0}{2\pi}\, \frac{1}{V} \sum_{\vec{k} \in L^3} (\not{k} + m)\, \delta(k^2 - m^2)\, \Theta(-k^0)\, e^{-ik(x-y)} \\ &= \frac{1}{2\pi V} \sum_{\vec{k} \in L^3} \frac{1}{2\,|k^0|}\, (\not{k} + m)\, e^{-ik(x-y)}\Big|_{k^0 = -\sqrt{|\vec{k}|^2 + m^2}}\,. \end{aligned} \tag{2.6.20}$$

Here the image of $(\not{k} + m)$ is two-dimensional; it is spanned by the two plane-wave solutions of the Dirac equation of momentum k with spin up and down, respectively.

Thus we can write the fermionic projector in analogy to (2.2.1) as

$$P^{\text{sea}}(x,y) \;=\; \sum_{\vec{k} \in L^3} \sum_{s=\pm 1} -|\Psi_{\vec{k}s}(x)\succ\!\prec \Psi_{\vec{k}s}(y)|\,, \tag{2.6.21}$$

where $\Psi_{\vec{k}s}$ are the suitably normalized negative-energy plane-wave solutions of the Dirac equation, and s denotes the two spin orientations. If an external field is present, it is still possible to decompose the fermionic projector similar to (2.6.21) into individual states. But clearly, each of these states is perturbed by $\mathcal{B}$; we denote these perturbed states by a tilde. The next theorem shows that the probability integral for these states is independent of the interaction and of the size of T^3.

THEOREM 2.6.2. **(probability integral)** *Under the assumptions of Theorem 2.6.1, every state $\tilde{\Psi}$ of the fermionic projector is normalized according to*

$$\int_{T^3} \prec\!\tilde{\Psi}|\gamma^0\, \tilde{\Psi}\succ\!(t,\vec{x})\, d\vec{x} \;=\; \frac{1}{2\pi}\,. \tag{2.6.22}$$

Proof. Since $\tilde{\Psi}$ is a solution of the Dirac equation $(i\partial\!\!\!/ + \mathcal{B} - mY - \mu \mathbb{1})\tilde{\Psi} = 0$, it follows from current conservation (see 1.2.16) that the probability integral (2.6.22) is time independent. Thus it suffices to compute it in the limits $t \to \pm\infty$, in which according to our decay assumptions on $\mathcal{B}$ the system is non-interacting. Since in the vacuum, the fermionic projector splits into a direct sum of Dirac seas, we may restrict attention to a single Dirac sea (2.6.21). Using that the probability integral is the same for both spin orientations,

$$\int_{T^3} \prec\!\Psi_{\vec{k}s}|\gamma^0\, \Psi_{\vec{k}s}\succ\!(t,\vec{x})\, d\vec{x} \;=\; \int_{T^3} \frac{1}{2} \sum_{s=\pm} \text{Tr}\left(\gamma^0\, |\Psi_{\vec{k}s}\succ\!\prec \Psi_{\vec{k}s}|\right) d\vec{x}\,,$$

and comparing with (2.6.20) gives

$$\begin{aligned} \int_{T^3} \prec\!\Psi_{\vec{k}s}|\gamma^0 \Psi_{\vec{k}s}\succ\!(t,\vec{x})\, d\vec{x} \;&=\; \frac{1}{4\pi\, V} \int_{T^3} \frac{1}{2k^0}\, \text{Tr}(\gamma^0\, (k\!\!\!/ + m))\big|_{k^0 = -\sqrt{|\vec{k}|^2+m^2}}\, d\vec{x} \\ &=\; \frac{1}{4\pi\, V} \int_{T^3} \frac{4k^0}{2k^0}\, d\vec{x} \;=\; \frac{1}{2\pi}\,. \end{aligned}$$

∎

Let us discuss what this result means for the states of the fermionic projector (2.6.13, 2.6.14). As pointed out in the paragraph of (2.6.8), the fermionic projector of the vacuum is composed for each $\vec{k} \in L^3$ of a continuum of states (2.6.8). However, if we choose the space-time points in the fixed time interval $-T < t < T$ and let $\delta \searrow 0$, we need not distinguish between the frequencies in (2.6.8) and obtain that only the discrete states with $\vec{k} \in L^3$, $k^0 = -\sqrt{|\vec{k}|^2 + m^2}$ are occupied (see the discussion after (2.6.8)). In the causal perturbation expansion, each of these states is perturbed, and thus also the interacting fermionic projector for small δ can be regarded as being composed of discrete states. We write in analogy to (2.6.21),

$$P(x,y) \;=\; \sum_a -|\tilde{\Psi}_a\succ\!\prec \tilde{\Psi}_a|\,,$$

where a runs over all the quantum number of the fermions. According to (2.6.13) and Theorem 1.5.5, the probability integral is

$$\int_{T^3} \prec \tilde{\Psi}_a \,|\, \gamma^0\, \tilde{\Psi}_a \succ (t, \vec{x})\, d\vec{x} \;=\; \frac{\delta}{2\pi}\,. \tag{2.6.23}$$

By substituting the formulas of the light-cone expansion of §2.5 into (2.6.13), one sees that the contributions of the light-cone expansion to the fermionic projector all involve at least one factor δ. Thus after rescaling P by a factor δ^{-1}, the probability integral (2.6.20) as well as the formulas of the light-cone expansion have a well-defined and non-trivial continuum limit. In particular, using that the particle and anti-particle states are to be normalized in accordance with the states of the sea, we can specify the normalization constant c_{norm} in (2.3.19). If the wave functions Ψ_k and Φ_l in (2.3.19) are normalized according to (1.2.15), we must choose

$$c_{\text{norm}} \;=\; -\frac{1}{2\pi}\,. \tag{2.6.24}$$

We finally remark that Theorem 1.5.5 can be generalized in a straightforward way to include a gravitational field, if (2.6.22) is replaced by (1.5.22), with $\mathcal{H}$ a space-like hypersurface with future-directed normal ν. However, we need to assume that the gravitational field decays at infinity in the sense that space-time is asymptotically flat and for $t \to \pm\infty$ goes over asymptotically to Minkowski space. In particular, realistic cosmological models like the Friedman-Robertson-Walker space-times are excluded. We do not expect that the large-scale structure of space-time should have an influence on the normalization[4], but this is an open problem which remains to be investigated.

[4]Online version: This picture has been confirmed by the paper arXiv:0901.0602 [math-ph].

CHAPTER 3

The Principle of the Fermionic Projector

In this chapter we introduce a new mathematical framework intended for the formulation of physical theories. We first generalize the notions of relativistic quantum mechanics and classical field theory in several construction steps. This will be done in a very intuitive way. The aim is to work out the essence of the underlying physical principles by dropping all additional and less important structures. This will lead us to a quite abstract mathematical framework. The "principle of the fermionic projector" states that the fundamental physical equations should be formulated within this framework. We conclude this chapter with a brief physical overview and discussion.

3.1. Connection between Local Gauge Freedom and the Measurability of Position and Time

In this section we give a possible explanation as to why local gauge freedom occurs in nature. This physical consideration will provide a formalism which will be the starting point for the constructions leading to the principle of the fermionic projector. We begin for simplicity with the example of the $U(1)$ gauge transformations of the magnetic field for a Schrödinger wave function Ψ in nonrelativistic quantum mechanics. Since it will be sufficient to consider the situation for fixed time, we only write out the spatial dependence of the wave function, $\Psi = \Psi(\vec{x})$ with $\vec{x} \in \mathbb{R}^3$. In the nonrelativistic and static limit, the gauge freedom of electrodynamics (1.4.1, 1.4.2) reduces to the transformations

$$\vec{A}(\vec{x}) \;\longrightarrow\; \vec{A}(\vec{x}) + \vec{\nabla}\Lambda(\vec{x}) \tag{3.1.1}$$

$$\Psi(\vec{x}) \;\longrightarrow\; e^{i\Lambda(\vec{x})}\, \Psi(\vec{x})\,, \tag{3.1.2}$$

where the so-called vector potential $\vec{A}$ consists of the three spatial components of the electromagnetic potential A. Similar to (1.4.4), we introduce the gauge-covariant derivative by

$$\vec{D} \;=\; \vec{\nabla} - i\vec{A}\,. \tag{3.1.3}$$

With the transformation (3.1.2) we can arbitrarily change the phase of the wave function Ψ at any point $\vec{x}$. This is consistent with the quantum mechanical interpretation of the wave function, according to which the phase of a wave function is not an observable quantity, only its absolute square $|\Psi(\vec{x})|^2$ has a physical meaning as the probability density of the particle. One can even go one step further and take the point of view that the inability to determine the local phase of a quantum mechanical wave function is the physical reason for the local gauge freedom (3.1.1, 3.1.2). Then the $U(1)$ gauge transformations of the magnetic field become a consequence of the principles of quantum mechanics. This argument becomes clearer when stated in more mathematical

terms: We consider on the Schrödinger wave functions the usual scalar product

$$<\Psi \,|\, \Phi> \;=\; \int_{\mathbb{R}^3} \overline{\Psi(\vec{x})}\, \Phi(\vec{x})\, d\vec{x}$$

and denote the corresponding Hilbert space by H. On H, the position operators $\vec{X}$ are given as the multiplication operators with the coordinate functions,

$$\vec{X}\, \Psi(\vec{x}) \;=\; \vec{x}\, \Psi(\vec{x})\,.$$

As it is common in quantum mechanics, we consider H as an abstract Hilbert space (i.e. we forget about the fact that H was introduced as a space of functions). Then the wave function $\Psi(\vec{x})$ corresponding to a vector $\Psi \in H$ is obtained by constructing a *position representation* of the Hilbert space. In bra/ket notation, this is done by choosing an "eigenvector basis" $|\vec{x}>$ of the position operators,

$$\vec{X}\, |\vec{x}> \;=\; \vec{x}\, |\vec{x}>\,, \qquad\qquad <\vec{x} \,|\, \vec{y}> \;=\; \delta^3(\vec{x}-\vec{y})\,, \tag{3.1.4}$$

and the wave function is then introduced by

$$\Psi(\vec{x}) \;=\; <\vec{x} \,|\, \Psi> \tag{3.1.5}$$

(we remark that the formal bra/ket notation can be made mathematically precise using spectral measures [**F3**]). The important point for us is that the "eigenvectors" $|\vec{x}>$ of the position operators are determined only up to a phase. Namely, the transformation

$$|\vec{x}> \;\longrightarrow\; \exp\left(-i\Lambda(\vec{x})\right) |\vec{x}> \tag{3.1.6}$$

leaves invariant the conditions (3.1.4) for the "eigenvector basis." If we substitute (3.1.6) into (3.1.5), we obtain precisely the transformation (3.1.2) of the wave function. The transformation properties of the gauge-covariant derivative (3.1.3) and of the gauge potentials in (3.1.1) follow from (3.1.2) if one assumes that the gauge-covariant derivatives $\vec{D}$ are operators on H (and thus do not depend on the representation of H as functions in position space). In physics, the operators $\vec{\pi} = -i\vec{D}$ are called the "canonical momentum operators."

The relation just described between the position representation of quantum mechanical states and the $U(1)$ gauge transformations of the magnetic field was noticed long ago. However, the idea of explaining local gauge freedom from quantum mechanical principles was not regarded as being of general significance. In particular, it was never extended to the relativistic setting or to more general gauge groups. The probable reason for this is that these generalizations are not quite straightforward; they make it necessary to formulate relativistic quantum mechanics in a particular way as follows. We again consider on the four-component Dirac spinors $(\Psi^\alpha(x))_{\alpha=1,\ldots,4}$ in Minkowski space the spin scalar product (1.2.8) and denote the vector space of all Dirac wave functions by H. Integrating the spin scalar product over space-time, we obtain an indefinite scalar product on H,

$$<\Psi \,|\, \Phi> \;=\; \int_{\mathbb{R}^4} \prec\!\Psi \,|\, \Phi\!\succ(x)\, d^4x\,. \tag{3.1.7}$$

Furthermore, we introduce on H time/position operators $(X^i)_{i=0,\ldots,3}$ by multiplication with the coordinate functions,

$$X^i\, \Psi(x) \;=\; x^i\, \Psi(x)\,.$$

We now consider $(H,\, <.|.>)$ as an abstract scalar product space. In order to construct a *time/position representation* of H, we must choose an "eigenvector basis"

of the time/position operators. Since the wave functions have four components, an "eigenvector basis" has in bra/ket notation the form $|x\alpha>$, $x \in \mathbb{R}^4$, $\alpha = 1, \ldots, 4$; it is characterized by the conditions

$$X^i\,|x\alpha> \;=\; x^i\,|x\alpha>\,, \qquad\qquad <x\alpha\,|\,y\beta> \;=\; s_\alpha\,\delta_{\alpha\beta}\,\delta^4(x-y) \tag{3.1.8}$$

with s_α as in (1.2.8). The wave function corresponding to a vector $\Psi \in H$ is defined by

$$\Psi^\alpha(x) \;=\; <x\alpha\,|\,\Psi>\,. \tag{3.1.9}$$

The conditions (3.1.8) determine the basis $|x\alpha>$ only up to local isometries of a scalar product of signature $(2,2)$, i.e. up to transformations of the form

$$|x\alpha> \;\longrightarrow\; \sum_{\beta=1}^{4} (U(x)^{-1})^\alpha_\beta\,|x\beta> \qquad \text{with} \qquad U(x) \in U(2,2)\,. \tag{3.1.10}$$

If we identify these transformations with gauge transformations and substitute into (3.1.9), we obtain local gauge freedom of the form

$$\Psi(x) \;\longrightarrow\; U(x)\,\Psi(x)\,. \tag{3.1.11}$$

Since gauge transformations correspond to changes of the "eigenvector basis" $|x\alpha>$, we also refer to $|x\alpha>$ as a *gauge*.

From the mathematical point of view, (3.1.8–3.1.10) is a straightforward generalization of (3.1.4–3.1.6) to the four-dimensional setting and four-component wave functions, taking into account that the spin scalar product has signature $(2,2)$. However, our construction departs from the usual description of physics, because the time operator X^0 is not commonly used in relativistic quantum mechanics and because the scalar product (3.1.7) is unconventional. In particular, one might object that the scalar product (3.1.7) may be infinite for physical states, because the time integral diverges. However, this is not a serious problem, which could be removed for example by considering the system in finite 4-volume and taking a suitable infinite volume limit. Furthermore, one should keep in mind that the scalar product (3.1.7) gives us the spin scalar product, and using the spin scalar product one can introduce the usual positive scalar product $(.|.)$ by integrating over a space-like hypersurface (see (1.2.17) or more generally (1.5.22)). Therefore, it causes no principal problems to consider $<.|.>$ instead of $(.|.)$ as the fundamental scalar product. We conclude that (3.1.7–3.1.10) is certainly an unconventional point of view, but it is nevertheless consistent and indeed mathematically equivalent to the usual description of relativistic quantum mechanics as outlined in §1.2.

The above explanation of local gauge freedom fits together nicely with our description of Dirac spinors in the gravitational field in §1.5: We let $(H, <.|.>)$ be the vector space of wave functions on a manifold M, endowed with the indefinite scalar product (1.5.23). For every coordinate system x^i we introduce the corresponding multiplication operators X^i. Considering H as an abstract vector space, the arbitrariness of the time/position representation of H again yields the local $U(2,2)$ gauge freedom (3.1.11). We thus obtain precisely the gauge transformations (1.5.2). In this way, (3.1.8–3.1.10) is not only consistent with all the constructions in §1.5, but it also gives a simple explanation for the gauge group $U(2,2)$.

The $U(2,2)$ gauge symmetry describes gravitation and electrodynamics, but it does not include the weak and strong interactions. In order to obtain additional gauge freedom, we must extend the gauge group. Since our gauge group is the isometry group

of the spin scalar product, this can be accomplished only by increasing the number of components of the wave functions. In general, one can take wave functions with $p+q$ components and a spin scalar product of signature (p,q),

$$\prec \Psi \,|\, \Phi \succ(x) \;=\; \sum_{\alpha=1}^{p+q} s_\alpha\, \Psi^\alpha(x)^*\, \Phi^\alpha(x) \qquad \text{with}$$
$$s_1 = \cdots = s_p = 1\,, \qquad s_{p+1} = \cdots = s_{p+q} = -1\,. \tag{3.1.12}$$

We call (p,q) the *spin dimension*. Repeating the above construction (3.1.7–3.1.9) for this spin scalar product yields local gauge freedom with gauge group $U(p,q)$. Unfortunately, it is not possible to introduce the Dirac operator in this generality. Therefore, we will always assume that the spin dimension is $(2N,2N)$ with $N \geq 1$. In this case, one can regard the $4N$ component wave functions as the direct sum of N Dirac spinors, exactly as we did in the general definition of the fermionic projector (2.3.1). Then our above argument yields the gauge group $U(2N,2N)$. The interaction can be described for example as in §2.3 by inserting a multiplication operator $\mathcal{B}$ into the Dirac operator (2.3.10) and taking the partial trace (2.3.20). More generally, one can modify the first order terms of the Dirac operator in analogy to Def. 1.5.1. Our concept is that the $U(2n,2n)$ gauge symmetry should be related to corresponding gauge potentials in the Dirac operator, and that this should, in the correct model, give rise to the gravitational, strong and electroweak gauge fields.

For clarity, we finally point out the differences of our approach to standard gauge theories. Usually, the gauge groups (e.g. the $SU(2)_{\mathrm{W}}$ or $SU(3)_{\mathrm{S}}$ in the standard model) act on separate indices of the wave functions (called the isospin and color indices, respectively). In contrast to this, our $U(2,2)$ gauge transformations simply act on the spinor index. In our generalization to higher spin dimension (3.1.12), we make no distinction between the spinor index and the index of the gauge fields; they are both combined in one index $\alpha = 1, \ldots, 4N$. In our setting, the gauge group and the coupling of the gauge fields to the Dirac particles are completely determined by the spin dimension. Compared to standard gauge theories, where the gauge groups and their couplings can be chosen arbitrarily, this is a strong restriction for the formulation of physical models.

3.2. Projection on Fermionic States

The fermionic projector was introduced in Chapter 2 in order to resolve the external field problem, and we used it to describe a general many-fermion system (2.3.19). We now discuss the concept of working with a "projector" in a more general context. A single Dirac particle is clearly described by its wave function $\Psi^\alpha(x) = <x\alpha|\Psi>$, or, in a gauge-independent way, by a vector $\Psi \in H$. Since the phase and normalization of Ψ have no physical significance, we prefer to describe the Dirac particle by the one-dimensional subspace $<\Psi> \equiv \{\lambda\Psi,\ \lambda \in \mathbb{C}\} \subset H$. Now consider the system of n Dirac particles, which occupy the one-particle states $\Psi_1, \ldots, \Psi_n \in H$. Generalizing the subspace $<\Psi>$ of the one-particle system, it seems natural to describe the many-particle system by the subspace $<\Psi_1, \ldots, \Psi_n> \subset H$ spanned by $\Psi_1, \ldots, \Psi_n$. We consider for simplicity only the generic case that this subspace is non-degenerate (i.e. there should be no vectors $0 \neq \Psi \in Y$ with $<\Psi|\Phi> = 0$ for all $\Phi \in Y$). Just as in positive definite scalar product spaces, every non-degenerate subspace $Y \subset H$ uniquely determines a projector P_Y on this subspace, characterized by the conditions

$P_Y^* = P_Y = P_Y^2$ and $\mathrm{Im}(P_Y) = Y$, where the star denotes the adjoint with respect to the scalar product $<.|.>$. Instead of working with the subspace $<\Psi_1, \ldots, \Psi_n> \subset H$, it is more convenient for us to consider the corresponding projector P,

$$P \;=\; P_{<\Psi_1,\ldots,\Psi_n>} \,.$$

We call P the *fermionic projector*. In this work we will always describe the Dirac particles of the system by a fermionic projector.

The concept of the fermionic projector departs from the usual description of a many-particle state by a vector of the fermionic Fock space (as introduced in §1.3). Let us discuss this difference in detail. In many-particle quantum mechanics, the system of Dirac particles $\Psi_1, \ldots, \Psi_n$ is described by the anti-symmetric product wave function

$$\Psi \;=\; \Psi_1 \wedge \cdots \wedge \Psi_n \,. \tag{3.2.1}$$

The wave functions of the form (3.2.1) are called n-particle Hartree-Fock states. They span the n-particle Fock space $F^n = \bigwedge^n H$. In the fermionic Fock space formalism, a quantum state is a linear combination of Hartree-Fock states, i.e. a vector of the Fock space $F = \bigoplus_{n=0}^\infty F^n$ (see §1.3 for details). In order to connect the fermionic projector with the Fock space formalism, we can associate to a projector P_Y on a subspace $Y = <\Psi_1, \ldots, \Psi_n> \subset H$ the wave function (3.2.1). This mapping clearly depends on the choice of the basis of Y. More precisely, choosing another basis $\Phi_i = \sum_{j=1}^n \kappa_i^j \Psi_j$, we have

$$\Phi_1 \wedge \cdots \wedge \Phi_n \;=\; \det(\kappa)\; \Psi_1 \wedge \cdots \wedge \Psi_n \,.$$

This shows that, due to the anti-symmetrization, the mapping is unique up to a complex factor. Therefore, with the mapping

$$P_{<\Psi_1,\ldots,\Psi_n>} \;\to\; <\Psi_1 \wedge \cdots \wedge \Psi_n> \subset F^n$$

we can associate to every projector a unique one-dimensional subspace of the Fock space. Since the image of this mapping is always a Hartree-Fock state, we obtain a one-to-one correspondence between the projectors P_Y on n-dimensional subspaces $Y \subset H$ and n-particle Hartree-Fock states. In this way, one sees that the description of a many-particle state with the fermionic projector is equivalent to using a Hartree-Fock state. With this correspondence, the formalism of the fermionic projector becomes a special case of the Fock space formalism, obtained by restricting to Hartree-Fock states. In particular, we conclude that the physical concepts behind fermionic Fock spaces, namely the Pauli Exclusion Principle and the fact that quantum mechanical particles are indistinguishable (see page 17), are also respected by the fermionic projector. However, we point out that the the fermionic projector is not mathematically equivalent to a state of the Fock space, because a vector of the Fock space can in general be represented only by a linear combination of Hartree-Fock states.

Let us analyze what this mathematical difference means physically. If nature is described by a fermionic projector, the joint wave function of all fermions of the Universe must be a Hartree-Fock state. However, this condition cannot be immediately verified in experiments, because measurements can never take into account all existing fermions. In all realistic situations, one must restrict the observations to a small subsystem of the Universe. As is worked out in Appendix A, the effective wave function of a subsystem need *not* be a Hartree-Fock state; it corresponds to an arbitrary vector of the Fock space of the subsystem, assuming that the number of particles of the whole system is sufficiently large. From this we conclude that the description of

the many-particle system with the fermionic projector is indeed physically equivalent to the Fock space formalism. For theoretical considerations, it must be taken into account that the fermionic projector merely corresponds to a Hartree-Fock state; for all practical purposes, however, one can just as well work with the whole Fock space.

We saw after (3.2.1) that the description of a many-particle state with the fermionic projector implies the Pauli Exclusion Principle. This can also be understood directly in a non-technical way as follows. For a given state $\Psi \in H$, we can form the projector $P_{<\Psi>}$ describing the one-particle state, but there is no projector which would correspond to a two-particle state (notice that the naive generalization $2P_{<\Psi>}$ is not a projector). More generally, every vector $\Psi \in H$ either lies in the image of P, $\Psi \in P(H)$, or it does not. Via these two conditions, the fermionic projector encodes for every state $\Psi \in H$ the occupation numbers 1 and 0, respectively, but it is impossible to describe higher occupation numbers. In this way, the fermionic projector naturally incorporates the Pauli Exclusion Principle in its formulation that each quantum mechanical state may be occupied by at most one fermion.

As explained at the end of §2.5, the fermionic projector contains all the information about the physical system in the sense that from a given fermionic projector one can uniquely reconstruct the fermionic states as well as the Dirac operator with interaction. Therefore, it is consistent to consider the fermionic projector as the basic object in space-time and to regard the Dirac operator merely as an auxiliary object which is useful in describing the interaction of the fermions via classical fields.

3.3. Discretization of Space-Time

The ultraviolet divergences of perturbative QFT indicate that the current description of physics should break down at very small distances. It is generally believed that the length scale where yet unknown physical effects should become important is given by the Planck length. Here we will assume that space-time consists on the Planck scale of discrete space-time points. The simplest way to discretize space-time would be to replace the space-time continuum by a four-dimensional lattice (as it is e.g. done in lattice gauge theories). In the following construction, we go much further and discretize space-time in a way where notions like "lattice spacing" and "neighboring lattice points" are given up. On the other hand, we will retain the principles of general relativity and our local gauge freedom.

We first consider the situation in a given coordinate system x^i in space-time[1]. For the discretization we replace the time/position operators X^i by mutually commuting operators with a *purely discrete spectrum*. We take the joint spectrum of these operators, i.e. the set

$$M \;=\; \{x \in \mathbb{R}^4 \,|\, \text{there is } u \in H \text{ with } X^i u = x^i u \text{ for all } i = 0, \ldots, 3\}\,,$$

as our discrete space-time points. We assume that the joint eigenspaces e_x of the X^i,

$$e_x \;=\; \{u \,|\, X^i u = x^i u \text{ for all } i = 0, \ldots, 3\}\,, \qquad\qquad x \in M\,,$$

[1]We assume for simplicity that the chart x^i describes all space-time. The generalization to a non-trivial space-time topology is done in a straightforward way by gluing together different charts; for details see [**F3**].

are $4N$-dimensional subspaces of H, on which the scalar product $<.|.>$ has the signature $(2N, 2N)$. Then we can choose a basis $|x\alpha>$, $x \in M$, $\alpha = 1, \ldots, 4N$ satisfying

$$\begin{aligned} X^i\,|x\alpha> &= x^i\,|x\alpha>\,, \qquad <x\alpha\,|\,y\beta> = s_\alpha\,\delta_{\alpha\beta}\,\delta_{xy} \qquad \text{with} \\ s_1 = \cdots = s_{2N} = 1\,, &\quad s_{2N+1} = \cdots = s_{4N} = -1\,. \end{aligned} \tag{3.3.1}$$

These relations differ from (3.1.8) only by the replacement $\delta^4(x-y) \to \delta_{xy}$. It is useful to introduce the projectors E_x on the eigenspaces e_x by

$$E_x = \sum_{\alpha=1}^{p+q} s_\alpha\,|x\alpha><x\alpha|\,; \tag{3.3.2}$$

they satisfy the relations

$$X^i\,E_x = x^i\,E_x \qquad \text{and} \tag{3.3.3}$$

$$E_x^* = E_x\,, \qquad E_x\,E_y = \delta_{xy}\,E_x\,, \qquad \sum_{x\in M} E_x = \mathbb{1}\,, \tag{3.3.4}$$

where the star denotes the adjoint with respect to the scalar product $<.|.>$ (these relations immediately follow from (3.3.1) and the fact that $|x\alpha>$ is a basis). The operators E_x are independent of the choice of the basis $|x\alpha>$, they are uniquely characterized by (3.3.3, 3.3.4) as the spectral projectors of the operators X^i.

If we change the coordinate system to $\tilde{x}^i = \tilde{x}^i(x)$, the discrete space-time points $M \subset \mathbb{R}^4$ are mapped to different points in $\mathbb{R}^4$, more precisely

$$\tilde{M} = \tilde{x}(M)\,, \qquad \tilde{E}_{\tilde{x}(x)} = E_x\,. \tag{3.3.5}$$

With such coordinate transformations, the relative position of the discrete space-time points in $\mathbb{R}^4$ can be arbitrarily changed. Taking general coordinate invariance seriously on the Planck scale, this is consistent only if we forget about the fact that M and $\tilde{M}$ are subsets of $\mathbb{R}^4$ and consider them merely as index sets for the spectral projectors. In other words, we give up the ordering of the discrete space-time points, which is inherited from the ambient vector space $\mathbb{R}^4$, and consider M and $\tilde{M}$ only as point sets. After this generalization, we can identify M with $\tilde{M}$ (via the equivalence relation $\tilde{x}(x) \simeq x$). According to (3.3.5), the spectral projectors $(E_p)_{p\in M}$ are then independent of the choice of coordinates.

We regard the projectors $(E_p)_{p\in M}$ as the basic objects describing space-time. The time/position operators can be deduced from them. Namely, every coordinate system yields an injection of the discrete space-time points

$$x\,:\,M \hookrightarrow \mathbb{R}^4\,, \tag{3.3.6}$$

and the corresponding time/position operators X^i can be written as

$$X^i = \sum_{p\in M} x^i(p)\,E_p\,. \tag{3.3.7}$$

Since every injection of the discrete space-time points into $\mathbb{R}^4$ can be realized by a suitable choice of coordinates (i.e. for every injection $\iota : M \hookrightarrow \mathbb{R}^4$ there is a chart x^i such that $x(M) = \iota(M)$), we can drop the condition that x is induced by a coordinate system. We can thus take for x in (3.3.6, 3.3.7) any embedding of M into $\mathbb{R}^4$.

Let us summarize the result of our construction. We shall describe space-time by an indefinite scalar product space $(H, <.|.>)$ and projectors $(E_p)_{p\in M}$ on H, where M is a (finite or countable) index set. The projectors E_p are characterized by the conditions

(3.3.4). Furthermore, we assume that the *spin dimension* is $(2N, 2N)$, i.e. $E_p(H) \subset H$ is for all $p \in M$ a subspace of signature $(2N, 2N)$. We call $(H, <.|.>, (E_p)_{p \in M})$ *discrete space-time*. The equivalence principle is taken into account via the freedom in choosing the embeddings (3.3.6, 3.3.7) of the discrete space-time points. Moreover, one can choose a basis $|p\alpha>$, $p \in M$, $\alpha = 1, \ldots, 4N$, of H satisfying the conditions

$$E_p\, |q\alpha> \;=\; \delta_{pq}\, |p\alpha>\,, \qquad\qquad <p\alpha \,|\, q\beta> \;=\; s_\alpha\, \delta_{\alpha\beta}\, \delta_{pq}$$

with s_α as in (3.3.1); such a basis is called a *gauge*. It is determined only up to transformations of the form

$$|p\alpha> \;\to\; \sum_{\beta=1}^{2N} (U(p)^{-1})^\alpha_\beta\, |p\beta> \quad \text{with} \quad U(p) \in U(2N, 2N)\,. \tag{3.3.8}$$

These are the local gauge transformations of discrete space-time.

3.4. The Principle of the Fermionic Projector

For the complete description of a physical system we must introduce additional objects in discrete space-time $(H, <.|.>, (E_p)_{p \in M})$. As mentioned at the end of §3.2, one can in the space-time continuum regard the fermionic projector as the basic physical object. Therefore, it seems promising to carry over the fermionic projector to discrete space-time. We introduce the *fermionic projector of discrete space-time* P as a projector acting on the vector space H of discrete space-time.

In analogy to the situation for the continuum, we expect that a physical system can be completely characterized by a fermionic projector in discrete space-time. At this stage, however, it is not at all clear whether this description makes any physical sense. In particular, it seems problematic that neither the Dirac equation nor the classical field equations can be formulated in or extended to discrete space-time; thus it becomes necessary to replace them by equations of completely different type. We take it as an ad-hoc postulate that this can actually be done; namely we assert

The Principle of the Fermionic Projector:
A physical system is completely described by the fermionic projector in discrete space-time. The physical equations should be formulated exclusively with the fermionic projector in discrete space-time, i.e. they must be stated in terms of the operators P and $(E_p)_{p \in M}$ on H.

Clearly, the validity and consequences of this postulate still need to be investigated; this is precisely the aim of the present work. The physical equations formulated with P and $(E_p)_{p \in M}$ are called the *equations of discrete space-time*.

3.5. A Variational Principle

Before coming to the general discussion of the principle of the fermionic projector, we want to give an example of a variational principle in discrete space-time. This is done to give the reader an idea of how one can formulate equations in discrete space-time. This example will serve as our model variational principle, and we will often come back to it. A more detailed motivation of our Lagrangian is given in Chapter 5.

Let us first discuss the general mathematical form of possible equations in discrete space-time. The operators P and $(E_p)_{p \in M}$ all have a very simple structure in that they are projectors acting on H. Therefore, it is not worth studying these operators separately; for physically promising equations, we must combine the projectors P

and $(E_p)_{p \in M}$ in a mathematically interesting way. Composite expressions in these operators can be manipulated using the idempotence of P and the relations (3.3.4) between the projectors $(E_p)_{p \in M}$: First of all, the identities $\sum_{p \in M} E_p = \mathbb{1}$ and $E_p^2 = E_p$ allow us to insert factors E_p into the formulas; e.g.

$$E_x\, P\, \Psi \;=\; E_x\, P \left(\sum_{y \in M} E_y \right) \Psi \;=\; \sum_{y \in M} (E_x\, P\, E_y)\, E_y\, \Psi\,.$$

Writing

$$P(x,y) \;\equiv\; E_x\, P\, E_y\,,$$

we obtain the identity

$$E_x\, (P\, \Psi) \;=\; \sum_{y \in M} P(x,y)\, E_y\, \Psi\,.$$

This representation of P by a sum over the discrete space-time points resembles the integral representation of an operator in the continuum with an integral kernel. Therefore, we call $P(x,y)$ the *discrete kernel* of the fermionic projector. The discrete kernel can be regarded as a canonical representation of the fermionic projector of discrete space-time, induced by the projectors $(E_p)_{p \in M}$. Now consider a general product of the operators P and $(E_p)_{p \in M}$. Using the relations $P^2 = P$ and $E_x\, E_y = \delta_{xy}\, E_x$, every operator product can be simplified to one with alternating factors P and E_p, i.e. to an operator product of the form

$$E_{x_1}\, P\, E_{x_2}\, P\, E_{x_3} \cdots E_{x_{n-1}}\, P\, E_{x_n} \qquad \text{with } x_j \in M. \tag{3.5.1}$$

Again using that $E_p^2 = E_p$, we can rewrite this product with the discrete kernel as

$$P(x_1, x_2)\, P(x_2, x_3) \;\cdots\; P(x_{n-1}, x_n)\,. \tag{3.5.2}$$

We conclude that the equations of discrete space-time should be formed of products of the discrete kernel, where the second argument of each factor must coincide with the first argument of the following factor. We refer to (3.5.2) as a *chain*.

In analogy to the Lagrangian formulation of classical field theory, we want to set up a variational principle. Our "action" should be a scalar functional depending on the operators P and E_p. Most scalar functionals on operators (like the trace or the determinant) can be applied only to endomorphisms (i.e. to operators which map a vector space into itself). The chain (3.5.2) is a mapping from the subspace $E_{x_n}(H) \subset H$ to $E_{x_1}(H)$. This makes it difficult to form a scalar, unless $x_1 = x_n$. Therefore, we will only consider *closed chains*

$$P(x, y_1)\, P(y_1, y_2) \cdots P(y_k, x) \;:\; E_x(H) \to E_x(H)\,.$$

In the simplest case $k = 0$, the closed chain degenerates to a single factor $P(x,x)$. This turns out to be too simple for the formulation of a physically interesting action, mainly because the light-cone structure of the fermionic projector (see §2.5) would then not enter the variational principle. Thus we are led to considering closed chains of two factors, i.e. to the operator product $P(x,y)P(y,x)$. Suppose that we are given a real-valued functional $\mathcal{L}$ on the endomorphisms of $E_x(H) \subset H$ (this will be discussed and specified below). Then $\mathcal{L}[P(x,y)\; P(y,x)]$ is a real function depending on two space-time arguments, and we get a scalar by summing over x and y. Therefore, we

take for our action S the ansatz

$$S \;=\; \sum_{x,y \in M} \mathcal{L}[P(x,y)\, P(y,x)]\,. \tag{3.5.3}$$

This ansatz is called a *two-point action*, and in analogy to classical field theory we call $\mathcal{L}$ the corresponding *Lagrangian*.

We shall now introduce a particular Lagrangian $\mathcal{L}$. The requirement which will lead us quite naturally to this Lagrangian is that $\mathcal{L}$ should be *positive*. Positivity of the action is desirable because it is a more convincing concept to look for a local minimum of the action than merely for a critical point of an action which is unbounded below.

Let us first consider how one can form a positive functional on $P(x,y)\,P(y,x)$. The closed chain $P(x,y)\,P(y,x)$ is an endomorphism of $E_x(H)$; we abbreviate it in what follows by A. In a given gauge, A is represented by a $4N \times 4N$ matrix. Under gauge transformations (3.3.8), this matrix transforms according to the adjoint representation,

$$A \;\to\; U(x)\, A\, U(x)^{-1}\,.$$

Furthermore, A is Hermitian on $E_x(H)$, i.e.

$$<A\,\Psi \,|\, \Phi> \;=\; <\Psi \,|\, A\,\Phi> \qquad\qquad \text{for } \Psi, \Phi \in E_x(H)\,, \tag{3.5.4}$$

or simply $A^* = A$. In positive definite scalar product spaces, the natural positive functional on operators is an operator norm, e.g. the Hilbert-Schmidt norm $\|B\|_2 = \mathrm{tr}(B^* B)^{\frac{1}{2}}$. In our setting, the situation is more difficult because our scalar product $<.|.>$ is indefinite on $E_x(H)$ (of signature $(2N, 2N)$). As a consequence, Hermitian matrices do not have the same nice properties as in positive definite scalar product spaces; in particular, the matrix A might have complex eigenvalues, and it is in general not even diagonalizable. Also, the operator product $A^* A$ need not be positive, so that we cannot introduce a Hilbert-Schmidt norm. In order to analyze the situation more systematically, we decompose the characteristic polynomial of A into linear factors

$$\det(\lambda - A) \;=\; \prod_{k=1}^{K} (\lambda - \lambda_k)^{n_k}\,. \tag{3.5.5}$$

This decomposition is useful because every functional on A can be expressed in terms of the roots and multiplicities of the characteristic polynomial; thus it is sufficient to consider the λ_k's and n_k's in what follows. Each root λ_k corresponds to an n_k-dimensional A-invariant subspace of $E_x(H)$, as one sees immediately from a Jordan representation of A. The roots λ_k may be complex. But since A is Hermitian (3.5.4), we know at least that the characteristic polynomial of A is real,

$$\overline{\det(\lambda - A)} \;=\; \det(\lambda - A) \qquad\qquad \text{for } \lambda \in \mathbb{R}.$$

This means that the complex conjugate of every root is again a root with the same multiplicity (i.e. for every λ_k there is a λ_l with $\overline{\lambda_k} = \lambda_l$ and $n_k = n_l$). The reality of the characteristic polynomial is verified in detail as follows. In a given gauge, we can form the transposed, complex conjugated matrix of A, denoted by $A^\dagger$. For clarity, we point out that $A^\dagger$ is *not* an endomorphism of $E_x(H)$, because it has the wrong behavior under gauge transformations (in particular, the trace $\mathrm{tr}(A^\dagger A)$ depends on the gauge and is thus ill-defined). Nevertheless, the matrix $A^\dagger$ is useful because we can write the adjoint of A in the form $A^* = S A^\dagger S$, where S is the spin signature matrix,

$S = \mathrm{diag}((s_\alpha)_{\alpha=1,\ldots,4N})$. Since $S^2 = \mathbb{1}$, and since the determinant is multiplicative, we conclude that for any real λ,

$$\begin{aligned}\overline{\det(\lambda - A)} &= \det(\lambda - A^\dagger) = \det(\lambda - S^2\, A^\dagger) \\ &= \det(\lambda - S A^\dagger S) = \det(\lambda - A^*) = \det(\lambda - A)\,.\end{aligned}$$

It is worth noting that every Lagrangian is symmetric in the two arguments x and y, as the following consideration shows. For any two quadratic matrices B and C, we choose ε not in the spectrum of C and set $C^\varepsilon = C - \varepsilon\mathbb{1}$. Taking the determinant of the relation $C^\varepsilon(BC^\varepsilon - \lambda) = (C^\varepsilon B - \lambda)C^\varepsilon$, we can use that the determinant is multiplicative and that $\det C^\varepsilon \neq 0$ to obtain the equation $\det(BC^\varepsilon - \lambda) = \det(C^\varepsilon B - \lambda)$. Since both determinants are continuous in ε, this equation holds even for all $\varepsilon \in \mathbb{R}$, proving the elementary identity

$$\det(BC - \lambda\mathbb{1}) = \det(CB - \lambda\mathbb{1})\,. \tag{3.5.6}$$

Applying this identity to the closed chain,

$$\det(P(x,y)\,P(y,x) - \lambda\mathbb{1}) = \det(P(y,x)\,P(x,y) - \lambda\mathbb{1})\,,$$

we find that the characteristic polynomial of the matrix A remains the same if the two arguments x and y are interchanged. Hence

$$\mathcal{L}[P(x,y)\,P(y,x)] = \mathcal{L}[P(y,x)\,P(x,y)]\,. \tag{3.5.7}$$

An obvious way to form a positive functional is to add up the absolute values of the roots, taking into account their multiplicities. We thus define the *spectral weight* $|A|$ of A by

$$|A| = \sum_{k=1}^K n_k\,|\lambda_k|\,. \tag{3.5.8}$$

This functional depends continuously on the λ_k, and also it behaves continuously when the roots of the characteristic polynomial degenerate and the multiplicities n_k change. Thus the spectral weight $|\,.\,|$ is a continuous functional. Furthermore, the spectral weight is zero if and only if the characteristic polynomial is trivial, $\det(\lambda - A) = \lambda^{4N}$. This is equivalent to A being nilpotent (i.e. $A^k = 0$ for some k). Thus, in contrast to an operator norm, the vanishing of the spectral weight does not imply that the operator is zero. On the other hand, it does not seem possible to define an operator norm in indefinite scalar product spaces, and therefore we must work instead with the spectral weight.

Using the spectral weight, one can write down many positive Lagrangians. The simplest choice would be $\mathcal{L}[A] = |A|$. Minimizing the corresponding action (3.5.3) yields a variational principle which attempts to make the absolute values of the roots $|\lambda_k|$ as small as possible. This turns out to be a too strong minimizing principle. It is preferable to formulate a variational principle which aspires to equalize the absolute values of all roots. This can be accomplished by combining the expressions $|A^2|$ and $|A|^2$. Namely, using that the sum of the multiplicities equals the dimension of the vector space, $\sum_{k=1}^K n_k = 4N$, the Schwarz inequality yields that

$$|A^2| = \sum_{k=1}^K n_k\,|\lambda_k|^2 \geq \frac{1}{4N}\left(\sum_{k=1}^K n_k\,|\lambda_k|\right)^2 = \frac{1}{4N}\,|A|^2\,,$$

and equality holds only if the absolute values of all roots are equal. Thus it is reasonable to minimize $|A^2|$, keeping $|A|^2$ fixed. This is our motivation for considering the two-point action:

$$\text{minimize} \quad S = \sum_{x,y \in M} \big|(P(x,y)\, P(y,x))^2\big| \tag{3.5.9}$$

under the constraint

$$T := \sum_{x,y \in M} |P(x,y)\, P(y,x)|^2 = \text{const}\,. \tag{3.5.10}$$

This is our model variational principle.

We next consider a stationary point of the action and derive the corresponding "Euler-Lagrange equations." For simplicity, we only consider the case that $P(x,y)P(y,x)$ can be diagonalized. This is the generic situation; the case of a non-diagonalizable matrix can be obtained from it by an approximation procedure. Having this in mind, we may assume that the endomorphism $A = P(x,y)\, P(y,x)$ has a spectral decomposition of the form

$$A = \sum_{k=1}^{K} \lambda_k\, F_k\,, \tag{3.5.11}$$

where λ_k are the roots in (3.5.5), and the F_k are operators mapping onto the corresponding eigenspaces (A, K, the λ_k, and the F_k clearly depend on x and y, but we will, for ease in notation, usually not write out this dependence). Since the underlying scalar product space is indefinite, the spectral decomposition (3.5.11) requires a brief explanation. Suppose that we choose a basis where A is diagonal. In this basis, the operators F_k are simply the diagonal matrices with diagonal entries 1 if the corresponding diagonal elements of A are λ_k, and 0 otherwise. Clearly, these operators map onto the eigenspaces and are orthonormal and complete, i.e.

$$A\, F_k = \lambda_k\, F_k\,, \quad F_k\, F_l = \delta_{kl}\, F_k \quad \text{and} \quad \sum_{k=1}^{K} F_k = \mathbb{1}_{E_x(H)}\,.$$

However, the F_k are in general *not* Hermitian (with respect to the spin scalar product). More precisely, taking the adjoint swaps the operators corresponding to complex conjugated eigenvalues,

$$F_k^* = F_l \qquad \text{when} \qquad \overline{\lambda_k} = \lambda_l\,. \tag{3.5.12}$$

These relations can be understood immediately because they ensure that the spectral decomposition (3.5.11) is Hermitian,

$$\left(\sum_{k=1}^{K} \lambda_k\, F_k\right)^* = \sum_{k=1}^{K} \overline{\lambda_k}\, F_k^* \overset{(3.5.12)}{=} \sum_{k=1}^{K} \lambda_k\, F_k\,.$$

Since the eigenvalues are in general complex, we can introduce a new matrix by taking the complex conjugate of the eigenvalues but leaving the spectral projectors unchanged,

$$\overline{A} = \sum_{k=1}^{K} \overline{\lambda_k}\, F_k \tag{3.5.13}$$

We refer to $\overline{A}$ as the *spectral adjoint* of A.

We now consider continuous variations $P(\tau)$ and $(E_p(\tau))_{p \in M}$, $-\varepsilon < \tau < \varepsilon$, of our operators. The structure of the operators must be respected by the variations;

this means that $P(\tau)$ should be a projector and that the relations (3.3.4) between the operators $(E_p)_{p \in M}$ should hold for all τ. Continuity of the variation implies that the rank of P and the signature of its image do not change. This implies that the variation of P can be realized by a unitary transformation

$$P(\tau) \;=\; U(\tau)\, P\, U(\tau)^{-1}\,, \tag{3.5.14}$$

where $U(\tau)$ is a unitary operator on H with $U(0) = \mathbb{1}$. Similarly, the variations of the projectors $(E_p)_{p \in M}$ are also unitary. From (3.3.4) we can conclude the stronger statement that the variations of all operators $(E_p)_{p \in M}$ can be realized by one unitary transformation, i.e.

$$E_p(\tau) \;=\; V(\tau)\, E_p\, V(\tau)^{-1}$$

with a unitary operator $V(\tau)$ and $V(0) = \mathbb{1}$. Since our action is invariant under unitary transformations of the vector space H, we can, instead of unitarily transforming both P and $(E_p)_{p \in M}$, just as well keep the $(E_p)_{p \in M}$ fixed and *vary only the fermionic projector* by (3.5.14). To first order in τ, this variation becomes

$$\delta P \;\equiv\; \frac{d}{d\tau} P(\tau)_{|\tau=0} \;=\; i\,[B,\, P]\,, \tag{3.5.15}$$

where $B = -iU'(0)$ is a Hermitian operator on H. We will only consider variations where B has *finite support* , i.e. where the kernel $B(x,y) \equiv E_x\, B\, E_y$ of B satisfies the condition

$$B(x,y) \;=\; 0 \qquad \text{except for } x, y \in N \subset M \text{ with } \#N \text{ finite.}$$

This condition can be regarded as the analogue of the assumption in the classical calculus of variations that the variation should have compact support.

Let us compute the variation of the action (3.5.9) (the constraint (3.5.10) will be considered afterwards). Writing out the action with the eigenvalues λ_k and multiplicities n_k, we obtain

$$S \;=\; \sum_{x,y \in M} \sum_{k=1}^{K} n_k\, |\lambda_k|^2\,.$$

The variation can be computed in perturbation theory to first order,

$$\begin{aligned} \delta S \;&=\; 2\,\mathrm{Re} \sum_{x,y \in M} \sum_{k=1}^{K} \overline{\lambda_k}\, \mathrm{tr}(F_k\, \delta A) \\ &=\; 2\,\mathrm{Re} \sum_{x,y \in M} \sum_{k=1}^{K} \overline{\lambda_k}\, \mathrm{tr}\left(F_k\, (\delta P(x,y)\, P(y,x) \;+\; P(x,y)\, \delta P(y,x))\right), \end{aligned}$$

where "tr" denotes the trace in the vector space H. Exchanging the names of x and y in the first summand in the trace and using cyclicity of the trace, this expression can be written as an operator product,

$$\delta S \;=\; 2\,\mathrm{Re}\; \mathrm{tr}(Q_1\, \delta P)\,, \tag{3.5.16}$$

where the kernel $Q_1(x,y) \equiv E_x\, Q_1\, E_y$ of Q_1 has the form

$$Q_1(x,y) \;=\; \left[\sum_{k=1}^{K} \overline{\lambda_k}\, F_k\right]_{xy} P(x,y) \;+\; P(x,y) \left[\sum_{k=1}^{K} \overline{\lambda_k}\, F_k\right]_{yx}, \tag{3.5.17}$$

and the subscripts "$_{xy}$" and "$_{yx}$" indicate that the corresponding brackets contain the spectral decomposition of the operators $P(x,y)\,P(y,x)$ and $P(y,x)\,P(x,y)$, respectively. Note that the trace in (3.5.16) is well-defined because the trace is actually taken only over a finite-dimensional subspace of H. At this point the following lemma is useful.

LEMMA 3.5.1. *Let B and C be two symmetric matrices and assume that their products $A := BC$ and $\tilde{A} := CB$ are both diagonalizable. Then they have the same eigenvalues $\lambda_1, \ldots, \lambda_K$ with the same multiplicities $n_1, \ldots, n_K$. The corresponding spectral projectors F_k and $\tilde{F}_k$ satisfy the relations*

$$F_k\, B \;=\; B\, \tilde{F}_k\,. \tag{3.5.18}$$

Proof. It immediately follows from (3.5.6) that the matrices A and $\tilde{A}$ have the same eigenvalues with the same multiplicities. For any λ not in the spectrum of A, we multiply the identity $B(CB-\lambda) = (BC-\lambda)B$ from the left and right by $(A-\lambda)^{-1}$ and $(\tilde{A}-\lambda)^{-1}$, respectively. This gives

$$(A-\lambda)^{-1}\, B \;=\; B\, (\tilde{A}-\lambda)^{-1}\,.$$

Integrating λ over a contour around any of the eigenvalues λ_k and using the Cauchy integral formulas

$$F_k \;=\; -\frac{1}{2\pi i} \oint_{\partial B_\epsilon(\lambda_k)} (A-\lambda)^{-1}\, d\lambda\,, \qquad \tilde{F}_k \;=\; -\frac{1}{2\pi i} \oint_{\partial B_\epsilon(\lambda_k)} (\tilde{A}-\lambda)^{-1}\, d\lambda\,,$$

we obtain (3.5.18). ∎

This lemma allows us to simplify (3.5.17) to

$$Q_1(x,y) \;=\; 2 \left[\sum_{k=1}^K \overline{\lambda_k}\, F_k \right]_{xy} P(x,y)\,. \tag{3.5.19}$$

A short straightforward computation using (3.5.12) and Lemma 3.5.18 shows that the operator Q_1 is Hermitian. Thus the trace in (3.5.16) is real, and we conclude that

$$\delta S \;=\; 2\,\mathrm{tr}(Q_1\, \delta P) \quad .$$

The variation of our constraint (3.5.10) can be computed similarly, and one gets

$$\begin{aligned} \delta T \;&=\; 2\,\mathrm{tr}(Q_2\, \delta P) \qquad \text{with} \\ Q_2(x,y) \;&=\; 2 \left[\Big(\sum_{l=1}^K n_l\, |\lambda_l| \Big) \sum_{k=1}^K \frac{\overline{\lambda_k}}{|\lambda_k|}\, F_k \right]_{xy} P(x,y)\,. \end{aligned}$$

Now consider a local minimum of the action. Handling the constraint with a Lagrange multiplier μ, we obtain the condition

$$0 \;=\; \delta S - \mu\, \delta T \;=\; 2\,\mathrm{tr}\left((Q_1 - \mu Q_2)\, \delta P\right) \overset{(3.5.15)}{=} 2i\,\mathrm{tr}\left((Q_1 - \mu Q_2)\, [B, P]\right).$$

Assume that the products $(Q_1 - \mu Q_2)\, P$ and $P\, (Q_1 - \mu Q_2)$ are well-defined operators. Since B has finite support, we can then cyclically commute the operators in the trace and obtain

$$0 \;=\; 4i\,\mathrm{tr}\left(B\, [P,\, Q_1 - \mu Q_2]\right).$$

Since B is arbitrary, we conclude that $[P,\, Q_1 - \mu Q_2] = 0$, where our notation with the commutator implicitly contains the condition that the involved operator products must be well-defined. Thus our *Euler-Lagrange equations* are the commutator equations

$$[P,\, Q] \;=\; 0 \qquad \text{with} \qquad Q(x,y) \;=\; 2\, C_{xy}\, P(x,y)\,, \tag{3.5.20}$$

$$C_{xy} \;=\; \sum_{k=1}^{K} \left[\overline{\lambda_k} \;-\; \mu\, \frac{\overline{\lambda_k}}{|\lambda_k|} \sum_{l=0}^{K} n_l\, |\lambda_l| \right]_{xy} F_k\,. \tag{3.5.21}$$

In the formula (3.5.21) for C_{xy}, we consider the spectral decomposition (3.5.5, 3.5.11) of the closed chain $P(x,y)\, P(y,x)$. The equations (3.5.20, 3.5.21) are the equations of discrete space-time corresponding to the variational principle (3.5.9, 3.5.10).

3.6. Discussion

In the previous sections the principle of the fermionic projector was introduced in a rather abstract mathematical way. Our constructions departed radically from the conventional formulation of physics, so much so that the precise relation between the principle of the fermionic projector and the notions of classical and quantum physics is not obvious. In order to clarify the situation, we now describe the general physical concept behind the principle of the fermionic projector and explain in words the connection to classical field theory, relativistic quantum mechanics and quantum field theory. Since we must anticipate results which will be worked out later, the description in this section is clearly not rigorous and is intended only to give a brief qualitative overview.

The constructions in §3.1 and §3.2 are merely a reformulation of classical field theory and relativistic quantum mechanics. Although they are an important preparation for the following construction steps, they do not by themselves have new physical implications. Therefore, we need not consider them here and begin by discussing the concept of discrete space-time of §3.3. With our definition of discrete space-time, the usual space-time continuum is given up and resolved into discrete space-time points. A-priori, the discrete space-time points are merely a point set without any relations (like for example the nearest-neighbor relation on a lattice) between them. Thus one may think of discrete space-time as a "disordered accumulation of isolated points." There exists no time parameter, nor does it make sense to speak of the "spatial distance" between the space-time points. Clearly, this concept of a pure point set is too general for a reasonable description of space-time. Namely, we introduced discrete space-time with the intention of discretizing the space-time continuum on the Planck scale. Thus, for systems which are large compared to the Planck length, the discrete nature of space-time should not be apparent. This means that discrete space-time should, in a certain *continuum limit*, go over to a Lorentzian manifold. However, since M is merely a point set, discrete space-time $(H,\, <.|.>,\, (E_p)_{p \in M})$ is symmetric under permutations of the space-time points. Taking a naive continuum limit would imply that the points of space-time could be arbitrarily exchanged, in clear contradiction to the topological and causal structure of a Lorentzian manifold.

In order to avoid this seeming inconsistency, one must keep in mind that we introduced an additional object space-time: the fermionic projector P. Via its discrete kernel $P(x,y)$, the fermionic projector yields relations between the discrete space-time points. Our idea is that the discrete kernel should provide all structures needed for a

reasonable continuum limit. In more detail, our concept is as follows. In the space-time continuum (see Chapter 2), the fermionic projector is built up of all quantum mechanical states of the fermionic particles of the system. Closely following Dirac's original concept, we describe the vacuum by the "sea" of all negative-energy states; systems with particles and anti-particles are obtained by occupying positive-energy states and removing states from the Dirac sea, respectively. The fermionic projector of the continuum completely characterizes the physical system. In particular, its integral kernel $P(x, y)$ is singular if and only if y lies on the light cone centered at x. In this way, the fermionic projector of the continuum encodes the causal, and thus also topological, structure of the underlying space-time. We have in mind that the fermionic projector of discrete space-time should, similar to a regularization on the Planck scale, approximate the fermionic projector of the continuum. This means that on a macroscopic scale (i.e. for systems comprising a very large number of space-time points), the fermionic projector of discrete space-time can, to good approximation, be identified with a fermionic projector of the continuum. Using the just-mentioned properties of the continuum kernel, we conclude that the discrete kernel induces on discrete space-time a structure which is well-approximated by a Lorentzian manifold. However, on the Planck scale (i.e. for systems involving only few space-time points), the discrete nature of space-time becomes manifest, and the notions of space, time and causality cease to exit.

The critical step for making this concept precise is the formulation of the physical equations intrinsically in discrete space-time. Let us describe in principle how this is supposed to work. In the continuum description of Chapter 2, the fermionic projector satisfies the Dirac equation (2.3.10); furthermore the potentials entering the Dirac equation obey classical field equations. As a consequence of these equations, the fermionic projector of the continuum is an object with very specific properties. Our idea is that, using the special form of the fermionic projector, it should be possible to restate the Dirac equation and classical field equations directly in terms of the fermionic projector. Thus we wish to formulate equations into which the fermionic projector enters as the basic object, and which are equivalent to, or a generalization of, both the Dirac equation and the classical field equations. It turns out that it is impossible to state equations of this type in the space-time continuum, because composite expressions in the fermionic projector are mathematically ill-defined. But one can formulate mathematically meaningful equations in discrete space-time, removing at the same time the ultraviolet problems of the continuum theory. The variational principle (3.5.9, 3.5.10) leading to the Euler-Lagrange equations (3.5.20, 3.5.21) is an example for such equations. Note that this variational principle and the corresponding Euler-Lagrange equations in discrete space-time are clearly not causal, but, for consistency with relativistic quantum mechanics and classical field theory, we demand that they should, in the continuum limit, reduce to local and causal equations (namely, to the Dirac and classical field equations). Since the fermionic projector is not an object which is commonly considered in physics, it is difficult to give an immediate physical interpretation for the equations of discrete space-time; only a detailed mathematical analysis can provide an understanding of the variational principle. If one wishes, one can regard the equations of discrete space-time as describing a direct particle-particle interaction between all the states of the fermionic projector. The collective interaction of the fermions of the Dirac sea with the additional particles and holes should, in the continuum limit, give rise to an effective interaction of fermions and anti-fermions via

classical fields. Ultimately, the collective particle-particle interaction should even give a microscopic justification for the appearance of a continuous space-time structure.

Let us now describe the relation to quantum field theory. Since the coupled Dirac and classical field equations, combined with the pair creation/annihilation of Dirac's hole theory, yield precisely the Feynman diagrams of QFT (see e.g. [**BD1**]), it is clear that all results of perturbative quantum field theory, in particular the high precision tests of QFT, are also respected by our ansatz (provided that the equations of discrete space-time have the correct continuum limit). Thus the only question is if the particular effects of quantized fields, like the Planck radiation and the photo electric effect, can be explained in our framework. The basic physical assumption behind Planck's radiation law is that the energy levels of an electromagnetic radiation mode do not take continuous values, but are quantized in steps of $E = \hbar\omega$. While the quantitative value $\hbar\omega$ of the energy steps can be understood via the quantum mechanical identification of energy and frequency (which is already used in classical Dirac theory), the crucial point of Planck's assumption lies in the occurrence of discrete energy levels. The photo electric effect, on the other hand, can be explained by a "discreteness" of the electromagnetic interaction: the electromagnetic wave tends not to transmit its energy continuously, but prefers to excite few atoms of the photographic material. We have the conception (which will, however, not be worked out in this book) that these different manifestations of "discreteness" should follow from the equations of discrete space-time if one goes beyond the approximation of an interaction via classical fields.

If this concept of explaining the effects of quantized fields from the equations of discrete space-time were correct, it would even have consequences for the interpretation of quantum mechanics. Namely, according to the statistical interpretation, quantum mechanical particles are point-like; the absolute value $|\Psi(\vec{x})|^2$ of the wave function gives the probability density for the particle to be at the position $\vec{x}$. Here, we could regard the wave function itself as the physical object; the particle character would come about merely as a consequence of the "discreteness" of the interaction of the wave function with e.g. the atoms of a photographic material. The loss of determinism could be explained naturally by the non-causality of the equations of discrete space-time.

We conclude that the principle of the fermionic projector raises quite fundamental questions on the structure of space-time, the nature of field quantization and the interpretation of quantum mechanics. Before entering the study of these general questions, however, it is most essential to establish a quantitative connection between the equations of discrete space-time and the Dirac and classical field equations. Namely, the principle of the fermionic projector can make physical sense only if it is consistent with classical field theory and relativistic quantum mechanics; thus it is of importance to first check this consistency. Even this comparatively simple limiting case is of highest physical interest. Indeed, the principle of the fermionic projector provides a very restrictive framework for the formulation of physical models; for example there is no freedom in choosing the gauge groups, the coupling of the gauge fields to the fermions, or the masses of the gauge bosons. This means that, if a connection could be established to relativistic quantum mechanics and classical field theory, the principle of the fermionic projector would give an explanation for the interactions observed in nature and would yield theoretical predictions for particle masses and coupling constants. We begin with this study in the next chapters.

CHAPTER 4

The Continuum Limit

According to the principle of the fermionic projector, we want to formulate physics with the fermionic projector P in discrete space-time $(H, <.|.>, (E_p)_{p \in M})$. In this chapter we will establish a mathematically sound connection between this description and the usual formulation of physics in a space-time continuum. More precisely, we will develop a general technique with which equations in discrete space-time, like for example the Euler-Lagrange equations (3.5.20, 3.5.21), can be analyzed within the framework of relativistic quantum mechanics and classical field theory. Our approach is based on the assumption that the fermionic projector of discrete space-time can be obtained from the fermionic projector of the continuum by a suitable regularization process on the Planck scale. The basic difficulty is that composite expressions in the fermionic projector (like in (3.5.20)) depend essentially on how the regularization is carried out; our task is to analyze this dependence in detail. We will show that, if we study the behavior close to the light cone, the dependence on the regularization simplifies considerably and can be described by a finite number of parameters. Taking these parameters as free parameters, we will end up with a well-defined effective continuum theory.

We point out that, since we deduce the fermionic projector of discrete space-time from the fermionic projector of the continuum, the causal and topological structure of the space-time continuum, as well as the Dirac equation and Dirac's hole theory, will enter our construction from the very beginning. Thus the continuum limit cannot give a justification or even derivation of these structures from the equations of discrete space-time (for such a justification one must go beyond the continuum limit; see §5.6 for a first attempt in this direction). The reason why it is nevertheless interesting to analyze the continuum limit is that we do not need to specify the classical potentials which enter the Dirac equation; in particular, we do not assume that they satisfy the classical field equations. Thus we can hope that an analysis of the equations of discrete space-time should give constraints for the classical potentials; this means physically that the equations of discrete space-time should in the continuum limit yield a quantitative description of the interaction of the Dirac particles via classical fields. This quantitative analysis of the continuum limit of interacting systems will be explained in Chapters 6–8.

For clarity we will mainly restrict attention to a fermionic projector consisting of one Dirac sea of mass m. The generalizations to systems of fermions with different masses and to chiral fermions (as introduced in §2.3) are given in §4.5. Having gauge fields in mind, which are in quantum field theory described by bosons, we often refer to the external potentials contained in the operator $\mathcal{B}$ in the Dirac equation (2.3.10) as *bosonic potentials* and the corresponding fields as *bosonic fields*.

4.1. The Method of Variable Regularization

Let us consider how one can get a relation between the continuum fermionic projector and the description of physics in discrete space-time. As discussed in §3.6, discrete space-time should for macroscopic systems go over to the usual space-time continuum. For consistency with relativistic quantum mechanics, the fermionic projector of discrete space-time should in this limit coincide with the continuum fermionic projector. Using furthermore that the discretization length should be of the order of the Planck length, we conclude that the fermionic projector of discrete space-time should correspond to a certain "regularization" of the continuum fermionic projector on the Planck scale. Thus it seems a physically reasonable method to construct the fermionic projector of discrete space-time from the fermionic projector of the continuum by a suitable regularization process on the Planck scale.

Regularizations of the continuum theory are also used in perturbative QFT in order to make the divergent Feynman diagrams finite. However, there is the following major difference between the regularizations used in QFT and our regularization of the fermionic projector. In contrast to QFT, where the regularization is merely a mathematical technique within the renormalization procedure, we here consider the regularized fermionic projector as the object describing the physical reality. The regularized fermionic projector should be a model for the fermionic projector of discrete space-time, which we consider as the basic physical object. As an important consequence, it is not inconsistent for us if the effective continuum theory depends on how the regularization is carried out. In this case, we must regularize in such a way that the regularized fermionic projector is a good microscopic approximation to the "physical" fermionic projector of discrete space-time; only such a regularization can yield the correct effective continuum theory. This concept of giving the regularization a physical significance clearly suffers from the shortcoming that we have no detailed information about the microscopic structure of the fermionic projector in discrete space-time, and thus we do not know how the correct regularization should look like. In order to deal with this problem, we shall consider a general class of regularizations. We will analyze in detail how the effective continuum theory depends on the regularization. Many quantities will depend sensitively on the regularization, so much so that they are undetermined and thus ill-defined in the continuum limit. However, certain quantities will be independent of the regularization and have a simple correspondence in the continuum theory; we call these quantities *macroscopic*. We will try to express the effective continuum theory purely in terms of macroscopic quantities. We cannot expect that the effective continuum theory will be completely independent of the regularization. But for a meaningful continuum limit, it must be possible to describe the dependence on the regularization by a small number of parameters, which we consider as empiric parameters modelling the unknown microscopic structure of discrete space-time. We refer to this general procedure for constructing the effective continuum theory as the *method of variable regularization*.

In order to illustrate the method of variable regularization, we mention an analogy to solid state physics. On the microscopic scale, a solid is composed of atoms, which interact with each other quantum mechanically. On the macroscopic scale, however, a solid can be regarded as a continuous material, described by macroscopic quantities like the density, the pressure, the conductivity, etc. The macroscopic quantities satisfy macroscopic physical equations like the equations of continuum mechanics, Ohm's law,

etc. Both the macroscopic characteristics of the solid and the macroscopic physical laws can, at least in principle, be derived microscopically from many-particle quantum mechanics. However, since the details of the microscopic system (e.g. the precise form of the electron wave functions) are usually not known, this derivation often does not completely determine the macroscopic physical equations. For example, it may happen that a macroscopic equation can be derived only up to a proportionality factor, which depends on unknown microscopic properties of the solid and is thus treated in the macroscopic theory as an empirical parameter. The physical picture behind the method of variable regularization is very similar to the physics of a solid, if one considers on the microscopic scale our description of physics in discrete space-time and takes as the macroscopic theory both relativistic quantum mechanics and classical field theory. Clearly, the concept of discrete space-time is more hypothetical than atomic physics because it cannot at the moment be verified directly in experiments. But we can nevertheless get indirect physical evidence for the principle of the fermionic projector by studying whether or not the method of variable regularization leads to interesting results for the continuum theory.

In the remainder of this section we will specify for which class of regularizations we shall apply the method of variable regularization. Our choice of the regularization scheme is an attempt to combine two different requirements. On one hand, we must ensure that the class of regularizations is large enough to clarify the dependence of the effective continuum theory on the regularization in sufficient detail; on the other hand, we must keep the technical effort on a reasonable level. Consider the integral kernel of the continuum fermionic projector (2.3.19, 2.5.45). Under the reasonable assumption that the fermionic wave functions Ψ_k and Φ_l are smooth, the projectors on the particle/anti-particle states in (2.3.19) are smooth in x and y. The non-causal low- and high-energy contributions P^{le} and P^{he} as well as the phase-inserted line integrals in (2.5.45) also depend smoothly on x and y. The factors $T^{(n)}$, however, have singularities and poles on the light cone (see (2.5.42) and (2.5.43)). Let us consider what would happen if we tried to formulate a variational principle similar to that in §3.5 with the continuum kernel (instead of the discrete kernel). The just-mentioned smooth terms in the kernel would not lead to any difficulties; we could just multiply them with each other when forming the closed chain $P(x,y)\,P(y,x)$, and the resulting smooth functions would influence the eigenvalues $\lambda_k(x,y)$ in (3.5.5) in a continuous way. However, the singularities of $T^{(n)}$ would cause severe mathematical problems because the multiplication of $T^{(n)}(x,y)$ with $T^{(n)}(y,x)$ leads to singularities which are ill-defined even in the distributional sense. For example, the naive product $P(x,y)\,P(y,x)$ would involve singularities of the form $\sim \delta'((y-x)^2)\,\delta((y-x)^2)$ and $\sim \delta((y-x)^2)^2$. This simple consideration shows why composite expressions in the fermionic projector make mathematical sense only after regularization. Furthermore, one sees that the regularization is merely needed to remove the singularities of $T^{(n)}$. Hence, it seems reasonable to regularize only the factors $T^{(n)}$ in (2.5.45), but to leave the fermionic wave functions Ψ_a, Φ_a as well as the bosonic potentials unchanged. This regularization method implies that the fermionic wave functions and the bosonic potentials are well-defined also for the regularized fermionic projector; using the notation of page 90, they are macroscopic quantities. Therefore, we call our method of only regularizing $T^{(n)}$ the *assumption of macroscopic potentials and wave functions*.

The assumption of macroscopic potentials and wave functions means physically that energy and momentum of all bosonic fields and of each particle/anti-particle of

the physical system should be small compared to the Planck energy. In other words, we exclude the case that the physical potentials and wave functions have oscillations or fluctuations on the Planck scale. Namely, such microscopic inhomogeneities could not be described by smooth functions in the continuum limit and are thus not taken into account by our regularization method. If, conversely, the potentials and wave functions are nearly constant on the Planck scale, the unregularized and the (no matter by which method) regularized quantities almost coincide, and it is thus a good approximation to work in the regularized fermionic projector with the unregularized potentials and wave functions.

According to the assumption of macroscopic potentials and wave functions, it remains to regularize the factors $T^{(n)}$ in (2.5.45). Recall that we constructed the distributions $T^{(n)}$ from the continuum kernel of the fermionic projector of the vacuum (2.2.1) via (2.5.39) and the expansion in the mass parameter (2.5.43). An essential step for getting a meaningful regularization scheme is to extend this construction to the case with regularization. Namely, this extension makes it sufficient to specify the regularization of the fermionic projector of the vacuum; we can then deduce the regularized $T^{(n)}$ and obtain, by substitution into (2.5.45), the regularized fermionic projector with interaction (if it were, on the contrary, impossible to derive the regularized $T^{(n)}$ from the regularized fermionic projector of the vacuum, the independent regularizations of all functions $T^{(n)}$, $n = -1, 0, 1, \ldots$, would involve so many free parameters that the effective continuum theory would be under-determined). Having in mind the extension of (2.5.39) and (2.5.43) to the case with regularization (which will be carried out in §4.5 and Appendix D), we now proceed to describe our regularization method for the fermionic projector of the vacuum. In the vacuum and for one Dirac sea, the kernel of the continuum fermionic projector $P(x, y)$ is given by the Fourier integral (2.2.2),

$$P(x,y) \;=\; \int \frac{d^4k}{(2\pi)^4}\, (\not{k} + m)\, \delta(k^2 - m^2)\, \Theta(-k^0)\, e^{-ik(x-y)}\,. \tag{4.1.1}$$

This distribution is invariant under translations in space-time, i.e. it depends only on the difference vector $y - x$. It seems natural and is most convenient to preserve the translation symmetry in the regularization. We thus assume that the kernel of the regularized fermionic projector of the vacuum, which we denote for simplicity again by $P(x, y)$, is translation invariant,

$$P(x,y) \;=\; P(y-x) \quad \text{for} \quad x, y \in M \subset \mathbb{R}^4\,. \tag{4.1.2}$$

We refer to (4.1.2) as a *homogeneous regularization of the vacuum*. Notice that the assumption (4.1.2) allows for both discrete and continuum regularizations. In the first case, the set M is taken to be a discrete subset of $\mathbb{R}^4$ (e.g. a lattice), whereas in the latter case, $M = \mathbb{R}^4$. According to our concept of discrete space-time, it seems preferable to work with discrete regularizations. But since continuous regularizations give the same results and are a bit easier to handle, it is worth considering them too. The assumption of a homogeneous regularization of the vacuum means physically that the inhomogeneities of the fermionic projector on the Planck scale should be irrelevant for the effective continuum theory. Since such microscopic inhomogeneities can, at least in special cases, be described by microscopic gravitational or gauge fields, this assumption is closely related to the assumption of macroscopic potentials and wave functions discussed above.

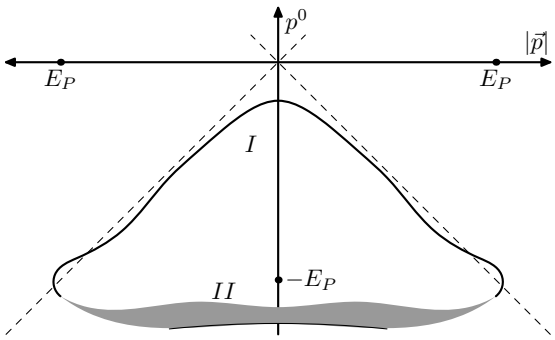

FIGURE 4.1. Example for $\hat{P}$, the regularized fermionic projector of the vacuum in momentum space.

Taking the Fourier transform in the variable $y-x$, we write (4.1.2) as the Fourier integral

$$P(x,y) \;=\; \int \frac{d^4p}{(2\pi)^4}\, \hat{P}(p)\, e^{-ip(x-y)} \tag{4.1.3}$$

with a distribution $\hat{P}$. If one considers a discrete regularization, $\hat{P}$ may be defined only in a bounded region of $\mathbb{R}^4$ (for a lattice regularization with lattice spacing d, for example, one can restrict the momenta to the "first Brillouin zone" $p \in (-\frac{\pi}{d}, \frac{\pi}{d})^4$). In this case, we extend $\hat{P}$ to all $\mathbb{R}^4$ by setting it to zero outside this bounded region. Although it will be of no relevance for what follows, one should keep in mind that for a discrete regularization, x and y take values only in the discrete set M. Let us briefly discuss the properties of the distribution $\hat{P}$. First of all, $P(x,y)$ should be the kernel of a Hermitian operator; this implies that $P(x,y)^* = P(y,x)$ and thus

$$\hat{P}(p)^* \;=\; \hat{P}(p) \qquad \text{for all } p \tag{4.1.4}$$

(where the star again denotes the adjoint with respect to the spin scalar product). For consistency with the continuum theory, the regularized kernel (4.1.3) should, for macroscopic systems, go over to the continuum kernel (4.1.1). Thus we know that $\hat{P}(p)$ should, for small energy-momentum p (i.e. when both the energy p^0 and the momentum $|\vec{p}|$ are small compared to the Planck energy), coincide with the distribution $(\not{p} + m)\, \delta(p^2 - m^2)\, \Theta(-p^0)$. This is illustrated in the example of Figure 4.1. In the region I close to the origin, $\hat{P}$ looks similar to a hyperbola on the lower mass shell. Furthermore, we know that $\hat{P}$ is a regularization on the Planck scale. This means that, in contrast to the integrand in (4.1.1), $\hat{P}$ should decay at infinity, at least so rapidly that the integral (4.1.3) is finite for all x and y. The length scale for this decay in momentum space should be of the order of the Planck energy $E_P = l_P^{-1}$. However, the precise form of $\hat{P}$ for large energy or momentum is completely arbitrary, as is indicated in Figure 4.1 by the "high energy region" II. This arbitrariness reflects our freedom in choosing the regularization.

We finally make an ansatz for $\hat{P}$ which seems general enough to include all relevant regularization effects and which will considerably simplify our analysis. According to (4.1.4), $\hat{P}(p)$ is a Hermitian 4×4 matrix and can thus be written as a real linear combination of the basis of the Dirac algebra $\mathbb{1}$, $i\rho$, $\rho\gamma^j$ and σ^{jk} (with the pseudoscalar matrix $\rho = i\gamma^0\gamma^1\gamma^2\gamma^3$ and the bilinear covariants $\sigma^{jk} = \frac{i}{2}[\gamma^j, \gamma^k]$). The integrand of the

continuum kernel (4.1.1) contains only vector and scalar components. It is reasonable to assume that the regularized kernel also contains no pseudoscalar and pseudovector components, because the regularization would otherwise break the symmetry under parity transformations. The inclusion of a bilinear component in $\hat{P}$, on the other hand, would cause technical complications but does not seem to give anything essentially new. Thus we make an ansatz where $\hat{P}$ is composed only of a vector and a scalar component, more precisely

$$\hat{P}(p) \;=\; (v_j(p)\,\gamma^j \;+\; \phi(p)\,\mathbb{1})\,f(p) \tag{4.1.5}$$

with a vector field v and a scalar field ϕ; f is a distribution. We also need to assume that $\hat{P}$ is reasonably regular and well-behaved; this will be specified in the following sections. We refer to the ansatz (4.1.5) as the assumption of a *vector-scalar structure* for the fermionic projector of the vacuum.

4.2. The Regularized Product $P(x,y)\,P(y,x)$ in the Vacuum

According to the method of variable regularization, we must analyze how the effective continuum theory depends on the choice of the regularization. We shall now consider this problem for the simplest composite expression in the fermionic projector, the closed chain $P(x,y)\,P(y,x)$ in the vacuum. The discussion of this example will explain why we need to analyze the fermionic projector on the light cone. Working out this concept mathematically will eventually lead us to the general formalism described in §4.5.

Using the Fourier representation (4.1.3), we can calculate the closed chain to be

$$\begin{aligned} P(x,y)\,P(y,x) &= \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4}\, \hat{P}(k_1)\,\hat{P}(k_2)\, e^{-i(k_1-k_2)(x-y)} \\ &= \int \frac{d^4p}{(2\pi)^4} \left[\int \frac{d^4q}{(2\pi)^4}\, \hat{P}(p+q)\,\hat{P}(q) \right] e^{-ip(x-y)}\,, \end{aligned} \tag{4.2.1}$$

where we introduced new integration variables $p = k_1 - k_2$ and $q = k_2$. Thus the Fourier transform of the closed chain is given by the convolution in the square brackets. This reveals the following basic problem. The convolution in the square bracket involves $\hat{P}$ for small and for large energy-momentum. Even when p is small, a large q leads to a contribution where both factors $\hat{P}(p+q)$ and $\hat{P}(q)$ are evaluated for large energy-momenta. If we look at the example of Figure 4.1, this means that (4.2.1) depends essentially on the behavior of $\hat{P}$ in the high-energy region II and can thus have an arbitrary value. More generally, we conclude that, since the form of $\hat{P}$ for large energy or momentum is unknown, the value of (4.2.1) is undetermined.

At first sight, it might seem confusing that the pointwise product $P(x,y)\,P(y,x)$ of the regularized fermionic projector should be undetermined, although the unregularized kernel (4.1.1) is, for $y-x$ away from the light cone, a smooth function, and so pointwise multiplication causes no difficulties. In order to explain the situation in a simple example, we briefly discuss the fermionic projector $\tilde{P}$ obtained by adding to P a plane wave,

$$\tilde{P}(x,y) \;=\; P(x,y) \;+\; e^{-ik(x-y)}\,\mathbb{1}\,.$$

If the energy or the momentum of the plane wave is of the order of the Planck energy, the plane wave is highly oscillatory in space-time. Such an oscillatory term is irrelevant on the macroscopic scale. Namely, if $\tilde{P}$ acts on a macroscopic function η, the oscillatory term is evaluated in the weak sense, and the resulting integral $\int \exp(iky)\,\eta(y)\,d^4y$ gives

almost zero because the contributions with opposite signs compensate each other. This "oscillation argument" can be made mathematically precise using integration by parts, e.g. in the case of high energy $k^0 \sim E_P$,

$$\int e^{iky}\, f(y)\, d^4y \;=\; -\frac{1}{ik^0} \int e^{iky}\, (\partial_t f)\, d^4y \;\sim\; \frac{1}{E_P}\,.$$

In the corresponding closed chain

$$\tilde{P}(x,y)\, \tilde{P}(y,x) \;=\; P(x,y)\, P(y,x) \;+\; P(x,y)\, e^{-ik(y-x)} \;+\; e^{-ik(x-y)}\, P(y,x) \;+\; \mathbb{1}\,,$$

the second and third summands are also oscillatory. In the last summand, however, the oscillations have dropped out, so that this term affects the macroscopic behavior of the closed chain. This elementary consideration illustrates why the unknown high-energy contribution to the fermionic projector makes it impossible to determine the closed chain pointwise. We remark that for very special regularizations, for example the regularization by convolution with a smooth "mollifier" function having compact support, the pointwise product makes sense away from the light cone and coincides approximately with the product of the unregularized kernels. But such regularizations seem too restrictive. We want to allow for the possibility that the fermionic projector describes non-trivial (yet unknown) high-energy effects. Therefore, the high-energy behavior of the fermionic projector should not be constrained by a too simple regularization method.

The fact that the product $P(x,y)\,P(y,x)$ is undetermined for fixed x and y does not imply that a pointwise analysis of the closed chain is mathematically or physically meaningless. But it means that a pointwise analysis would essentially involve the unknown high-energy behavior of $\hat{P}$; at present this is a problem out of reach. Therefore, our strategy is to find a method for evaluating the closed chain in a way where the high-energy behavior of $\hat{P}$ becomes so unimportant that the dependence on the regularization can be described in a simple way. We hope that this method will lead us to a certain limiting case in which the equations of discrete space-time become manageable.

The simplest method to avoid the pointwise analysis is to evaluate the closed chain in the weak sense. The Fourier representation (4.2.1) yields that

$$\int P(x,y)\, P(y,x)\, \eta(x)\, d^4x \;=\; \int \frac{d^4p}{(2\pi)^4}\, \hat{\eta}(p) \left[\int \frac{d^4q}{(2\pi)^4}\, \hat{P}(p+q)\, \hat{P}(q) \right], \qquad (4.2.2)$$

where $\hat{\eta}$ is the Fourier transform of a smooth function η. For macroscopic η (i.e. a function which is nearly constant on the Planck scale), the function $\hat{\eta}(p)$ is localized in a small neighborhood of $p = 0$ and has rapid decay. Thus, exactly as (4.2.1), the integral (4.2.2) depends on the form of $\hat{P}$ for large energy-momentum. Hence this type of weak analysis is not helpful. In order to find a better method, we consider again the Fourier integral (4.1.3) in the example of Figure 4.1. We want to find a regime for $y - x$ where the "low energy region" I plays an important role, whereas the region II is irrelevant. This can be accomplished only by exploiting the special form of $\hat{P}$ in the low-energy region as follows. The hyperbola of the lower mass shell in region I comes asymptotically close to the cone $C = \{p^2 = 0\}$. If we choose a vector $(y-x) \neq 0$ on the light cone $L = \{(y-x)^2 = 0\}$, then the hypersurface $\mathcal{H} = \{p \,|\, p(y-x) = 0\}$ is null and thus tangential to the cone C. This means that for all states on the hyperbola which are close to the straight line $C \cap \mathcal{H}$, the exponential in (4.1.3) is approximately one. Hence all these states are "in phase" and thus yield a large contribution to the Fourier

integral (4.1.3). The states in the high-energy region II, however, are not in phase; they will give only a small contribution to (4.1.3), at least when the vector $(y-x) \in L$ is large, so that the exponential in (4.1.3) is highly oscillatory on the scale $p \sim E_P$. This qualitative argument shows that by considering the fermionic projector on the light cone, one can filter out information on the behavior of $\hat{P}$ in the neighborhood of a straight line along the cone C. This should enable us to analyze the states on the lower mass shell without being affected too much by the unknown high-energy behavior of $\hat{P}$. We point out that if $P(x,y)$ depends only on the behavior of $\hat{P}$ close to the cone C, then the same is immediately true for composite expressions like the product $P(x,y)\, P(y,x)$. Thus restricting our analysis to the light cone should simplify the dependence on the regularization considerably, also for composite expressions like the closed chain. Our program for the remainder of this chapter is to make this qualitative argument mathematically precise and to quantify it in increasing generality.

4.3. The Regularized Vacuum on the Light Cone, Scalar Component

For simplicity we begin the analysis on the light cone for the scalar component of (4.1.5), i.e. we consider the case

$$\hat{P}(p) \;=\; \phi(p)\, f(p) \tag{4.3.1}$$

(the vector component will be treated in the next section). We can assume that the spatial component of the vector $y-x$ in (4.1.3) points in the direction of the x-axis of our Cartesian coordinate system, i.e. $y-x = (t, r, 0, 0)$ with $r > 0$. Choosing cylindrical coordinates ω, k, ρ and φ in momentum space, defined by $p = (\omega, \vec{p})$ and $\vec{p} = (k,\ \rho\cos\varphi,\ \rho\sin\varphi)$, the Fourier integral (4.1.3) takes the form

$$P(x,y) \;=\; \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dk \int_0^{\infty} \rho\, d\rho \int_0^{2\pi} d\varphi\; \hat{P}(\omega,k,\rho,\varphi)\; e^{i\omega t - ikr}\,. \tag{4.3.2}$$

Since the exponential factor in this formula is independent of ρ and φ, we can write the fermionic projector as the two-dimensional Fourier transform

$$P(x,y) \;=\; 2 \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dk\; h(\omega,k)\; e^{i\omega t - ikr} \tag{4.3.3}$$

of a function h defined by

$$h(\omega,k) \;=\; \frac{1}{2\,(2\pi)^4} \int_0^{\infty} \rho\, d\rho \int_0^{2\pi} d\varphi\; (\phi\, f)(\omega,k,\rho,\varphi)\,. \tag{4.3.4}$$

We want to analyze $P(x,y)$ close to the light cone $(y-x)^2 = 0$ away from the origin $y = x$. Without loss of generality, we can restrict attention to the upper light cone $t = r$. Thus we are interested in the region $t \approx r > 0$. The "light-cone coordinates"

$$s \;=\; \frac{1}{2}\,(t-r)\,, \qquad\qquad l \;=\; \frac{1}{2}\,(t+r) \tag{4.3.5}$$

are well-suited to this region, because the "small" variable s vanishes for $t = r$, whereas the "large" variable l is positive and non-zero. Introducing also the associated momenta

$$u \;=\; -k-\omega\,, \qquad\qquad v \;=\; k-\omega\,, \tag{4.3.6}$$

we can write the fermionic projector as

$$P(s,l) \;=\; \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv\; h(u,v)\; e^{-i(us+vl)}\,. \tag{4.3.7}$$

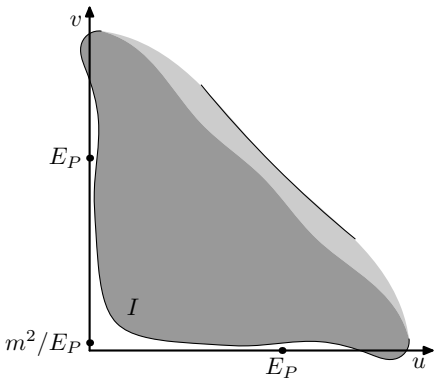

FIGURE 4.2. Example for $h(u,v)$, the reduced two-dimensional distribution.

Let us briefly discuss the qualitative form of the function h, (4.3.4). According to the continuum kernel (4.1.1), the scalar component (4.3.1) should, for energy and momentum small compared to the Planck energy E_P, go over to the δ-distribution on the lower mass shell $\hat{P} = m\,\delta(p^2 - m^2)\,\Theta(-p^0)$. In this limit, the integral (4.3.4) can be evaluated to be

$$\begin{aligned} h &= \frac{m}{2\,(2\pi)^4} \int_0^\infty \rho\, d\rho \int_0^{2\pi} d\varphi\; \delta(\omega^2 - k^2 - \rho^2 - m^2)\,\Theta(-\omega) \\ &= \frac{m}{4\,(2\pi)^3}\,\Theta(\omega^2 - k^2 - m^2)\,\Theta(-\omega) \;=\; \frac{m}{32\pi^3}\,\Theta(uv - m^2)\,\Theta(u)\,; \end{aligned} \tag{4.3.8}$$

thus integrating out ρ and φ yields a constant function in the interior of the two-dimensional "lower mass shell" $\omega^2 - k^2 = m^2$, $\omega < 0$. From this we conclude that for $u, v \ll E_P$, $h(u,v)$ should have a discontinuity along the hyperbola $\{uv = m^2,\, u > 0\}$, be zero below (i.e. for $uv < m^2$) and be nearly constant above. Furthermore, we know that h decays at infinity on the scale of the Planck energy. Similar to our discussion of $\hat{P}$ after (4.1.4), the precise form of h for large energy or momentum is completely arbitrary. The function $h(u,v)$ corresponding to the example of Figure 4.1 is shown in Figure 4.2. The two branches of the hyperbola asymptotic to the u and v axes are labeled by "A" and "B," respectively.

It is instructive to consider the energy scales of our system. The scale for high energies is clearly given by the Planck energy E_P. The relevant low-energy scale, on the other hand, is m^2/E_P (it is zero for massless fermions). This is because the hyperbola $uv = m^2$ comes as close to the v-axis as as $v \sim m^2/E_P$ before leaving the low-energy region. These two energy scales are also marked in Figure 4.2. Since we want to analyze the situation close to the light cone, we choose the "small" light-cone parameter s on the Planck scale, i.e.

$$s \sim E_P^{-1} \qquad \text{or} \qquad s < E_P^{-1}\,. \tag{4.3.9}$$

The "large" light-cone parameter l, on the other hand, is non-zero and thus yields a third energy scale. We shall always choose this scale between the two extremal energy scales, more precisely

$$\frac{1}{E_P} \;\ll\; l \;<\; l_{\max} \;\ll\; \frac{E_P}{m^2}\,. \tag{4.3.10}$$

The parameter $l_{\max}$ was introduced here in order to avoid l being chosen too large. Namely, we will always regard l as being small compared to the length scales of macroscopic physics (a reasonable value for $l_{\max}$ would e.g. be the Fermi length). One should keep in mind that the quotient of the two fundamental energy scales is in all physical situations extremely large; namely $E_P^2/m^2 \gg 10^{35}$. Thus the constraints (4.3.10) can be easily satisfied and still leave us the freedom to vary l on many orders of magnitude.

In the remainder of this section we shall evaluate the Fourier integral (4.3.7) using the scales (4.3.9) and (4.3.10). In preparation, we discuss and specify the function $h(u,v)$ for fixed u, also denoted by $h_u(v)$. As one sees in Figure 4.2, h_u will in general not be continuous. More precisely, in the example of Figure 4.2, h_u has a discontinuous "jump" from zero to a finite value on the hyperbola (and its extension to the high-energy region) and maybe has a second jump to zero for large v (e.g. on line "a"). For simplicity, we assume that h_u is always of this general form, i.e.

$$h_u(v) \;=\; \begin{cases} 0 & \text{for } v < \alpha_u \text{ or } v > \beta_u \\ \text{smooth} & \text{for } \alpha_u \leq v \leq \beta_u \end{cases} \tag{4.3.11}$$

with parameters $\alpha_u < \beta_u$. The case of less than two discontinuities can be obtained from (4.3.11) by setting $h_u(\alpha_u)$ or $h_u(\beta_u)$ equal to zero, or alternatively by moving the position of the discontinuities α_u or β_u to infinity. We remark that the discontinuity at $v = \beta_u$ will become irrelevant later; it is here included only to illustrate why the behavior of the fermionic projector on the light cone is independent of many regularization details. Without regularization, $h_u(v)$ is for $v \geq \alpha_u$ a constant function, (4.3.8). Thus the v-dependence of $h_u(v)$ for $\alpha_u \leq v \leq \beta_u$ is merely a consequence of the regularization, and it is therefore reasonable to assume that the v-derivatives of $h_u(v)$ scale in inverse powers of the regularization length E_P. More precisely, we demand that there is a constant $c_1 \ll l E_P$ with

$$|h_u^{(n)}(v)| \;\leq\; \left(\frac{c_1}{E_P}\right)^n \max|h_u| \qquad \text{for } \alpha_u \leq v \leq \beta_u\,, \tag{4.3.12}$$

where the derivatives at $v = \alpha_u$ and β_u are understood as the right- and left-sided limits, respectively. This regularity condition is typically satisfied for polynomial, exponential and trigonometric functions, but it excludes small-scale fluctuations of h_u. Clearly, we could also consider a more general ansatz for h_u with more than two discontinuities or weaker regularity assumptions. But this does not seem to be the point because all interesting effects, namely the influence of discontinuities for small and large v, as well as of smooth regions, can already be studied in the setting (4.3.11, 4.3.12).

Let us analyze the v-integral of the Fourier transform (4.3.7),

$$P_u(l) \;:=\; \int_{-\infty}^{\infty} h_u(v)\, e^{-ivl}\, dv\,. \tag{4.3.13}$$

According to the first inequality in (4.3.10), the exponential factor in (4.3.13) is highly oscillatory on the scale $v \sim E_P$. Thus we can expect that the smooth component of h_u gives only a small contribution to the integral (4.3.13), so that the discontinuities at α_u and β_u play the dominant role. In order to make this picture mathematically

precise, we iteratively integrate in (4.3.13) K times by parts,

$$\begin{aligned} P_u(l) &= \int_{\alpha_u}^{\beta_u} h_u(v)\, e^{-ivl}\, dv \;=\; -\frac{1}{il} \int_{\alpha_u}^{\beta_u} dv\; h_u(v)\, \frac{d}{dv} e^{-ivl} \\ &= -\frac{1}{il}\, h_u(v)\, e^{-ivl}\Big|_{\alpha_u}^{\beta_u} + \frac{1}{il} \int_{\alpha_u}^{\beta_u} h_u'(v)\, e^{-ivl}\, dl \;=\; \cdots \;= \\ &= -\frac{1}{il} \sum_{n=0}^{K-1} \left(\frac{1}{il}\right)^n h_u^{(n)}(v)\, e^{-ivl}\Big|_{\alpha_u}^{\beta_u} + \left(\frac{1}{il}\right)^K \int_{\alpha_u}^{\beta_u} h_u^{(K)}(v)\, e^{-ivl}\, dl\,. \end{aligned} \tag{4.3.14}$$

If we bound all summands in (4.3.14) using the first inequality in (4.3.10) and the regularity condition (4.3.12), each v-derivative appears in combination with a power of l^{-1}, and this gives a factor $c_1/(lE_P) \ll 1$. Thus we can in the limit $K \to \infty$ drop the integral in (4.3.14) and obtain

$$P_u(l) \;=\; -\frac{1}{il} \sum_{n=0}^{\infty} \left(\frac{1}{il}\right)^n h_u^{(n)}(v)\, e^{-ivl}\Big|_{\alpha_u}^{\beta_u}\,. \tag{4.3.15}$$

This expansion converges, and its summands decay like $(c_1/(lE_P))^n$.

Using (4.3.13), we can write the Fourier transform (4.3.7) as

$$P(s,l) \;=\; \int_{-\infty}^{\infty} P_u(l)\, e^{-ius}\, du\,. \tag{4.3.16}$$

Notice that, apart from the constraints (4.3.10), the "large" variable l can be freely chosen. We want to study the functional dependence of (4.3.16) on the parameter l. In preparation, we consider an integral of the general form

$$\int_a^b f(u)\, e^{-i\gamma(u)\, l}\, du\,, \tag{4.3.17}$$

where we assume that $(u, \gamma(u))$ is a curve in the high-energy region, more precisely $\gamma \sim E_P$. Assume furthermore that γ is monotone with $|\gamma'| \sim 1$ and that $(b-a) \sim E_P$. By transforming the integration variable, we can then write (4.3.17) as the Fourier integral

$$\int_{\gamma(a)}^{\gamma(b)} f\, |\gamma'|^{-1}\, e^{-i\gamma l}\, d\gamma\,. \tag{4.3.18}$$

If the function $f\, |\gamma'|^{-1}$ is smooth, its Fourier transform (4.3.18) has rapid decay in the variable l. Under the stronger assumption that $f\, |\gamma'|^{-1}$ varies on the scale E_P, we conclude that the length scale for this rapid decay is of the order $l \sim E_P^{-1}$. As a consequence, the rapid decay can be detected even under the constraint $l < l_{\max}$ imposed by (4.3.10), and we say that (4.3.18) has *rapid decay in* l. The reader who feels uncomfortable with this informal definition can immediately make this notion mathematically precise by an integration by parts argument similar to (4.3.14) imposing for $f\, |\gamma'|^{-1}$ a condition of type (4.3.12). The precise mathematical meaning of rapid decay in l for the integral (4.3.17) is that for every integer k there should be constants $c \sim 1$ and $l_{\min} \ll l_{\max}$ such that for all $l \in (l_{\min}, l_{\max})$,

$$\int_a^b f(u)\, e^{-i\gamma(u)\, l}\, du \;\leq\; c\, (lE_P)^{-k} \int_a^b |f(u)|\, du\,.$$

We return to the analysis of the integral (4.3.16). The boundary terms of (4.3.15) at β_u yield contributions to $P(s,l)$ of the form

$$-\left(\frac{1}{il}\right)^{n+1} \int_{-\infty}^{\infty} h_u^{(n)}(\beta_u)\, e^{-i\beta_u l - ius}\, du\,. \tag{4.3.19}$$

Recall that the points (u, β_u) are in the high-energy region (in the example of Figure 4.2, these points lie on curve "a"). According to (4.3.9), the length scale for the oscillations of the factor $\exp(-ius)$ is $u \sim E_P$. Under the reasonable assumption that β_u is monotone and that the functions $|\beta'(u)|^{-1}$ and $h_u^{(n)}(\beta_u)$ vary on the scale E_P, the integral (4.3.19) is of the form (4.3.18), and the above consideration yields that (4.3.19) has rapid decay in l. We remark that this argument could be extended to the case where β_u has extremal points (basically because the extrema give contributions only for isolated momenta u and thus can be shown to be negligible), but we will not go into this here. Having established rapid decay in l for (4.3.19), it remains to consider the boundary terms in (4.3.19) at α_u, more precisely

$$\begin{aligned} P(s,l) &= \sum_{n=0}^{\infty} \left(\frac{1}{il}\right)^{n+1} \int_{-\infty}^{\infty} h_u^{(n)}(\alpha_u)\, e^{-i\alpha_u l - ius}\, du \\ &\quad + (\text{rapid decay in } l)\,. \end{aligned} \tag{4.3.20}$$

We cannot again apply our "oscillation argument" after (4.3.17), because α_u tends asymptotically to zero on branch "A" of the hyperbola (see Figure 4.2), so that the factor $\exp(-i\alpha_u l)$ is non-oscillating in this region. We expand this factor in a Taylor series,

$$P(s,l) = \sum_{n,k=0}^{\infty} \frac{1}{k!}\, (il)^{k-n-1} \int_{-\infty}^{\infty} h_u^{(n)}(\alpha_u)\, (-\alpha_u)^k\, e^{-ius}\, du\,. \tag{4.3.21}$$

In the region where $l\alpha_u \not\ll 1$, this expansion might seem problematic and requires a brief explanation. First of all, α_u becomes large near $u = 0$ (on branch "B" of the hyperbola in Figure 4.2). In the case without regularization, the power expansion of the factor $\exp(-i\alpha_u\, l)$ corresponds to an expansion in the mass parameter (recall that in this case, $\alpha_u = m^2/u$ according to (4.3.8)), and in (4.3.21) it would lead to a singularity of the integrand at the origin. Indeed, this difficulty is a special case of the logarithmic mass problem which was mentioned in §2.5 and was resolved by working with the "regularized" distribution T_a^{reg} (2.5.42). Using these results, the behavior of the unregularized $P(s,l)$ for small momenta $u \ll E_P$ is well understood. Our oscillation argument after (4.3.17) yields that the regularization for $u \ll E_P$ (i.e. the form of the extension of branch "B" of the hyperbola to the high-energy region) affects $P(s,l)$ merely by rapidly decaying terms. Thus it is sufficient to consider here the integrand in (4.3.21) away from the origin $u = 0$. When combined with the results in §2.5, our analysis will immediately yield a complete description of the regularized fermionic projector near the light cone. Furthermore, the function α_u might become large for $u \sim E_P$, and this is a more subtle point. One way of justifying (4.3.21) would be to simply assume that $l_{\max}\alpha_u \ll 1$ along the whole extension of branch "A" to the high-energy region. A more general method would be to split up the curve (u, α_u) in the high-energy region $u \sim E_P$ into one branch where the expansion (4.3.21) is justified and another branch where our oscillation argument after (4.3.17) applies. The intermediate region $l\alpha_u \sim 1$, where none of the two methods can be

used, is generically so small that it can be neglected. In order to keep our analysis reasonably simple, we here assume that α_u is sufficiently small away from the origin, more precisely

$$\alpha_u \;<\; \alpha_{\max} \;\ll\; l_{\max}^{-1} \qquad \text{for } u \sim E_P. \tag{4.3.22}$$

For a fixed value of $k-n$, all summands in (4.3.21) have the same l-dependence. Let us compare the relative size of these terms. According to our regularity assumption (4.3.12), the derivatives of h scale like $h_u^{(n)} \sim E_P^{-n}$. Using the bound (4.3.22), we conclude that, for a fixed power of l, the summands in (4.3.21) decrease like $(\alpha_{\max}/E_P)^n$. Thus it is a very good approximation to drop the summands for large n. At first sight, it might seem admissible to take into account only the first summand $n=0$. But the situation is not quite so simple. For example, it may happen that, when restricted to the curve (u, α_u), the function $h(u,v)$ is so small that the summands for $n=0$ in (4.3.21) are indeed not dominant. More generally, we need to know that for some $n_0 \geq 0$, the function $h_u^{(n_0)}(\alpha_u)$ is really of the order given in (4.3.12), i.e.

$$|h_u^{(n_0)}(\alpha_u)| \;\geq\; c \left(\frac{c_1}{E_P}\right)^{n_0} \max |h_u| \qquad \text{for } u \sim E_P \tag{4.3.23}$$

with a positive constant c which is of the order one. If this condition is satisfied, we may neglect all summands for $n > n_0$, and collecting the terms in powers of l, we conclude that

$$\begin{aligned} &P(s,l) \\ &\quad= \frac{1}{(il)^{n_0+1}} \sum_{k=0}^{\infty} (-il)^k \sum_{n=\max(n_0-k,0)}^{n_0} \frac{(-1)^{n_0-n}}{(k-n_0+n)!} \int_{-\infty}^{\infty} h_u^{(n)}(\alpha_u)\, \alpha_u^{k-n_0+n}\, e^{-ius}\, du \\ &\qquad + \sum_{n=n_0+1}^{\infty} \frac{1}{(il)^{n+1}} \int_{-\infty}^{\infty} h_u^{(n)}(\alpha_u)\, e^{-ius}\, du \;+\; (\text{rapid decay in } l) \\ &\qquad + (\text{higher orders in } (\alpha_{\max}/E_P)\,. \end{aligned} \tag{4.3.24}$$

We point out that, according to (4.3.22),

$$\alpha_{\max}/E_P \ll (l_{\max} E_P)^{-1}\,,$$

and this explains why we disregard the higher orders in $\alpha_{\max}/E_P$. In our case, the function h_u has in the low-energy region according to (4.3.8) the form $h_u(\alpha_u) = m/(32\pi^3)\,\Theta(u)$. Hence it is natural to assume that (4.3.23) is satisfied for $n_0 = 0$. Introducing the shorter notation

$$h(u) \;:=\; h_u(\alpha(u))\,, \qquad h^{[n]}(u) \;:=\; h_u^{(n)}(\alpha_u)\,, \qquad \alpha(u) \;:=\; \alpha_u\,, \tag{4.3.25}$$

we have thus derived the following result.

Expansion of the scalar component: *Close to the light cone (4.3.9, 4.3.10), the*

scalar component (4.3.1) of the fermionic projector of the vacuum has the expansion

$$P(s,l) \;=\; \frac{1}{il} \sum_{k=0}^{\infty} \frac{(-il)^k}{k!} \int_{-\infty}^{\infty} h\, \alpha^k\, e^{-ius}\, du \tag{4.3.26}$$

$$+ \sum_{n=1}^{\infty} \frac{1}{(il)^{n+1}} \int_{-\infty}^{\infty} h^{[n]}\, e^{-ius}\, du \tag{4.3.27}$$

$$+ \,(\text{rapid decay in } l) \;+\; (\text{higher orders in } (\alpha_{\max}/E_P) \tag{4.3.28}$$

with suitable regularization functions h, $h^{[n]}$ and α. In the low-energy region $u \ll E_P$, the regularization functions are

$$h(u) \;=\; \frac{m}{32\pi^3}\, \Theta(u)\,, \qquad h^{[n]}(u) \;=\; 0\,, \qquad \alpha(u) \;=\; \alpha_u \;=\; \frac{m^2}{u}\,. \tag{4.3.29}$$

In this expansion, the l-dependence is written out similar to a Laurent expansion. The main simplification compared to our earlier Fourier representation (4.1.3) is that the dependence on the regularization is now described by functions of only one variable, denoted by h, $h^{[n]}$ and α. In composite expressions in $P(s,l)$, we will typically get convolutions of these functions; such one-dimensional convolutions are convenient and can be easily analyzed. The simplification to one-dimensional regularization functions became possible because many details of the regularization affect only the contribution with rapid decay in l, which we do not consider here. Notice that the summands in (4.3.26) and (4.3.27) decay like $(l\,\alpha_{\max})^k/k! \ll (l/l_{\max})^k/k!$ and $(lE_P)^{-n}$, respectively. In the low-energy limit (4.3.29), the expansion (4.3.26) goes over to a power series in m^2, and we thus refer to (4.3.26) as the *mass expansion*. In the mass expansion, the regularization is described by only two functions h and α. The series (4.3.27), on the other hand, is a pure regularization effect and is thus called the *regularization expansion*. It involves an infinite number of regularization functions $h^{[n]}$. Accordingly, we will use the notions of mass and regularization expansions also for other expansions of type (4.3.24).

In the expansion (4.3.24), the fermionic projector is described exclusively in terms of the function $h(u,v)$ in a neighborhood of the discontinuity along the curve (u, α_u). Let us go back to the definition of h, (4.3.4), and consider what this result means for the regularized fermionic projector in momentum space (4.3.1). In the case without regularization (4.3.8), we saw that integrating out the cylindrical coordinates ρ and φ yields a discontinuity of h whenever the 2-plane $(\omega, k) = \text{const}$ meets and is tangential to the hyperboloid $\omega^2 - k^2 - \rho^2 = m^2$. This picture is true in the general case in the sense that the discontinuity of h can be associated to a contribution to $\hat{P}$ which describes a hypersurface in four-dimensional momentum space. The simplest way to recover the discontinuity of h when integrating out the cylindrical coordinates would be to choose $\hat{P}$ of the form (4.3.1) with a function ϕ and the spherically symmetric distribution $f = \delta(|\vec{p}| - \omega - \alpha(-|\vec{p}| - \omega))$. Since spherically symmetric regularizations seem too restrictive, it is preferable to describe the discontinuity of h more generally by a contribution to $\hat{P}$ of the form

$$\phi(\vec{p})\; \delta(\omega - \Omega(\vec{p}))\,, \tag{4.3.30}$$

which is singular on the hypersurface $\omega = \Omega(\vec{p})$. For small momentum $|\vec{p}| \ll E_P$, the surface should clearly go over to the mass shell given by $\Omega = -\sqrt{|\vec{p}|^2 + m^2}$ and $\phi = m/|2\Omega|$; also, it is reasonable to assume that ϕ and Ω are smooth and sufficiently

regular. This consideration shows that for the behavior of the fermionic projector on the light cone (4.3.24), the essential role is played by states lying on a hypersurface. We refer to these one-particle states as the *surface states* of the fermionic projector of the vacuum. This result seems physically convincing because the surface states naturally generalize the states on the lower mass shell known from relativistic quantum mechanics. By integrating out the cylindrical coordinates for the ansatz (4.3.30), one can express the regularization functions $h_u^{(n)}$ in (4.3.24) in terms of ϕ and the geometry of the hypersurface. But we point out that, in contrast to the just discussed discontinuity of h, the partial derivatives of h depend also on states other than surface states. For example, a contribution to $\hat{P}$ of the form $b(\omega, \vec{p})\, \Theta(\omega - \Omega(\vec{p}))$ with Ω as in (4.3.30) and a smooth function b has a discontinuity on the surface Ω and affects all the regularization functions $h_u^{(n)}$ for $n \geq 1$ (as one verifies by a short computation). Thinking of the decomposition of the fermionic projector into the one-particle states, such non-surface contributions would consist of a large number of states and would thus make it necessary to introduce many additional fermions into our system. It does not seem quite reasonable or appropriate to considerably increase the number of particles of the system with the only purpose of having more freedom for the derivative terms of h in (4.3.24). It seems easiest and physically most convincing to assume that all the regularization functions in (4.3.24) come about as a consequence of surface states. We refer to this assumption as the *restriction to surface states*. It is of no relevance for the scalar component (4.3.26, 4.3.28), but it will yield an important relation between the regularization functions for the vector component in the next section. To avoid confusion, we point out that the restriction to surface states clearly does not imply that $\hat{P}$ is of the form (4.3.30). It imposes a condition only on the behavior of $\hat{P}$ in a neighborhood of our hypersurface; namely that the only distributional or non-regular contribution to $\hat{P}$ in this neighborhood should be the hypersurface itself.

For clarity, we finally review our assumptions on the regularization. Our first assumption was that the function $h(u, v)$ has, for every fixed u, at most two discontinuities at $\alpha(u)$ and $\beta(u)$, and is sufficiently regular otherwise (4.3.12). Furthermore, the function $\beta(u)$ had to be monotone and again sufficiently regular. For the function $\alpha(u)$, we assumed that (4.3.22) holds. Since h is obtained from $\hat{P}$, (4.3.1), by integrating out the cylindrical coordinates (4.3.4), these assumptions implicitly pose conditions on the fermionic projector of the vacuum. Although they could clearly be weakened with more mathematical effort, these conditions seem sufficiently general for the moment. In order to understand this better, one should realize that integrating out the cylindrical coordinates does generically (i.e. unless there are singularities parallel to the plane $(\omega, k) = \text{const}$) improve the regularity. The restriction to the generic case is in most situations justified by the fact that the direction $y - x$ and the coordinate system in (4.3.2) can be freely chosen. Using the above assumptions on $h(u, v)$, we showed that the dominant contribution to the fermionic projector on the light cone comes from states on a hypersurface in four-dimensional momentum space. With the "restriction to surface states" we assumed finally that the behavior on the light cone (4.3.24) is completely characterized by these states.

4.4. The Regularized Vacuum on the Light Cone, Vector Component

We shall now extend the previous analysis to the vector component in (4.1.5). More precisely, we will analyze the Fourier integral (4.1.3) for

$$\hat{P}(p) \;=\; v_j(p)\, \gamma^j\, f(p) \tag{4.4.1}$$

close to the light cone. We again choose light-cone coordinates (s, l, x_2, x_3) with $y-x = (s, l, 0, 0)$ (s and l are given by (4.3.5), while x_2 and x_3 are Cartesian coordinates in the orthogonal complement of the sl-plane). The associated momenta are denoted by $p = (u, v, p_2, p_3)$ with u and v according to (4.3.6). As in (4.3.3), we integrate out the coordinates perpendicular to u and v,

$$h_j(u,v) \;:=\; \frac{1}{2\,(2\pi)^4} \int_{-\infty}^{\infty} dp_2 \int_{-\infty}^{\infty} dp_3\; (v_j\, f)(u, v, p_2, p_3)\,, \tag{4.4.2}$$

and obtain a representation of the fermionic projector involving two-dimensional Fourier integrals

$$P(s,l) \;=\; \gamma^j\, P_j(s,l)$$

with

$$P_j(s,l) \;:=\; \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv\; h_j(u,v)\, e^{-i(us+vl)}\,. \tag{4.4.3}$$

The tensor indices in (4.4.2) and (4.4.3) refer to the coordinate system (s, l, x_2, x_3). For clarity, we denote the range of the indices by $j = s, l, 2, 3$; thus

$$\gamma^s \;=\; \frac{1}{2}\,(\gamma^0 - \gamma^1)\,, \qquad \gamma^l \;=\; \frac{1}{2}\,(\gamma^0 + \gamma^1)\,, \tag{4.4.4}$$

where $\gamma^0, \ldots, \gamma^3$ are the usual Dirac matrices of Minkowski space. According to the continuum kernel (4.1.1), $\hat{P}$ has in the case without regularization the form $\hat{P} = \not{p}\, \delta(p^2 - m^2)\, \Theta(-p^0)$ and h_j can be computed similar to (4.3.8) to be

$$\gamma^j\, h_j(u,v) \;=\; \frac{1}{32\pi^3}(-u\gamma^s - v\gamma^l)\, \Theta(uv - m^2)\, \Theta(u)\,. \tag{4.4.5}$$

This limiting case specifies the regularized $h_j(u,v)$ for small energy-momentum $u, v \ll E_P$. In order to keep the form of the functions h_j in the high-energy region sufficiently general, we merely assume in what follows that the h_j satisfy all the conditions we considered for the function h in the previous section (see the summary in the last paragraph of §4.3). Our main result is the following.

Expansion of the vector component: *Close to the light cone (4.3.9, 4.3.10), the vector component (4.4.1) of the fermionic projector of the vacuum has the expansion* $P = \gamma^j P_j$ *with*

$$\begin{aligned} P_s(s,l) &= \frac{1}{il} \sum_{k=0}^{\infty} \frac{(-il)^k}{k!} \int_{-\infty}^{\infty} -u\, g_s\ \alpha^k\, e^{-ius}\, du \\ &\quad + \sum_{n=1}^{\infty} \frac{1}{(il)^{n+1}} \int_{-\infty}^{\infty} -u\, g_s^{[n]}\, e^{-ius}\, du \\ &\quad + (\text{rapid decay in } l) + (\text{higher orders in } (\alpha_{\max}/E_P) \qquad (4.4.6) \\ P_l(s,l) &= \frac{1}{(il)^2} \sum_{k=0}^{\infty} \frac{(-il)^k}{k!} \int_{-\infty}^{\infty} \left[(k-1)\, \alpha^k \;+\; k\, \frac{b}{u}\, \alpha^{k-1} \right] g_l\, e^{-ius}\, du \\ &\quad + \sum_{n=1}^{\infty} \frac{1}{(il)^{n+2}} \int_{-\infty}^{\infty} -(n+1)\, g_l^{[n]}\, e^{-ius}\, du \\ &\quad + (\text{rapid decay in } l) + (\text{higher orders in } (\alpha_{\max}/E_P) \qquad (4.4.7) \\ P_{2\!/\!3}(s,l) &= \frac{1}{(il)^2} \sum_{k=0}^{\infty} \frac{(-il)^k}{k!} \int_{-\infty}^{\infty} \left[\alpha^k \;+\; k\, \frac{b_{2\!/\!3}}{u}\, \alpha^{k-1} \right] g_{2\!/\!3}\, e^{-ius}\, du \\ &\quad + \sum_{n=1}^{\infty} \frac{1}{(il)^{n+2}} \int_{-\infty}^{\infty} g_{2\!/\!3}^{[n]}\, e^{-ius}\, du \\ &\quad + (\text{rapid decay in } l) + (\text{higher orders in } (\alpha_{\max}/E_P) \qquad (4.4.8) \end{aligned}$$

and suitable regularization functions g_j, $g_j^{[n]}$, b, $b_{2\!/\!3}$ and the mass regularization function α as in (4.3.26, 4.3.29). In the low energy region $u \ll E_P$, the regularization functions have the form

$$g_s(u) = \frac{1}{32\pi^3}\, \Theta(u)\,, \qquad g_s^{[n]}(u) = 0 \qquad (4.4.9)$$

$$g_l(u) = \frac{1}{32\pi^3}\, \Theta(u)\,, \qquad g_l^{[n]}(u) = b(u) = 0 \qquad (4.4.10)$$

$$g_{2\!/\!3}(u) = g_{2\!/\!3}(u) = b_{2\!/\!3}(u) = 0\,. \qquad (4.4.11)$$

Before entering the derivation, we briefly discuss these formulas. To this end, we consider the situation where, like in the case without regularization, the vector $v(p)$ in (4.4.1) points into the direction p. In this case we can write the vector component as

$$\hat{P}(p) = p_j \gamma^j\, (\phi f)(p)\,, \qquad (4.4.12)$$

where (ϕf) has the form of the scalar component considered in §4.3. Since multiplication in momentum space corresponds to differentiation in position space, we obtain for (4.4.3)

$$P(s,l) = -i \left(\gamma^s \frac{\partial}{\partial s} + \gamma^l \frac{\partial}{\partial l} + \gamma^2 \frac{\partial}{\partial x^2} + \gamma^3 \frac{\partial}{\partial x^3} \right) P_{\text{scalar}}(s,l)\,,$$

where P_{scalar} is the scalar component (4.3.7) with h as in (4.3.4). We now substitute for P_{scalar} the expansion on the light cone (4.3.26–4.3.28) and carry out the partial derivatives. For the s- and l-components, this gives exactly the expansions (4.4.6, 4.4.7) with

$$g_s = g_l = h\,, \qquad g_s^{[n]} = g_l^{[n]} = h^{[n]}\,, \qquad b = 0\,. \qquad (4.4.13)$$

For the components $j = 2, 3$, the calculation of the partial derivatives is not quite so straightforward because the expansion of the scalar component (4.3.26–4.3.28) was carried out for fixed x_2 and x_3. Nevertheless, one can deduce also the expansion (4.4.8) from (4.3.26–4.3.28) if one considers x_2 and x_3 as parameters of the regularization functions h, $h^{[n]}$ and α, and differentiates through, keeping in mind that differentiation yields a factor $1/l$ (to get the scaling dimensions right). In this way, the simple example (4.4.12) explains the general structure of the expansions (4.4.6–4.4.8). We point out that the regularization function b vanishes identically in (4.4.13). This means that b is non-zero only when the direction of the vector field v is modified by the regularization. Thinking in terms of the decomposition into the one-particle states, we refer to this regularization effect as the *shear of the surface states*.

We shall now derive the expansions (4.4.6–4.4.8). Since the Fourier integrals in (4.4.3) are of the form (4.3.7), they have the expansion (4.3.24), valid close to the light cone (4.3.9, 4.3.10). It remains to determine the parameter n_0 in (4.3.24). We consider the components $j = s, l, 2$ and 3 separately. According to (4.4.5), the function h_s in the low-energy region looks similar to the hyperbola depicted in Figure 4.2. The main difference to the low-energy behavior of the scalar component (4.3.8) is the additional factor u in h_s which grows linearly along branch "A" of the hyperbola. Thus in the low-energy region away from the origin,

$$(h_s)_u(\alpha_u) \;\sim\; E_P \qquad \text{and} \qquad \max_{v \in (0, E_P)} |(h_s)_u(v)| \;\sim\; E_P \,. \tag{4.4.14}$$

Hence it is natural to assume that h_s satisfies the bound (4.3.23) with $n_0 = 0$. Because of the linearly growing factor u in the low-energy region, it is convenient to write the regularization functions in the form

$$(h_s)_u(\alpha_u) \;=:\; -u\, g_s(u) \,, \qquad (h_s)^{(n)}_u(\alpha_u) \;=:\; -u\, g_s^{[n]}(u) \tag{4.4.15}$$

with suitable functions g_s and $g_s^{[n]}$ (this can be done because, as explained after (4.3.21), close to the origin $u = 0$, we can work with the unregularized fermionic projector). This yields the expansion (4.4.6). According to (4.4.5) and (4.4.15), the regularization functions have the low-energy limit (4.4.9). For the l-component, the situation is much different. According to (4.4.5), the function h_l in the low-energy limit has the form

$$h_l(u, v) \;=\; -\frac{1}{32\pi^3}\, v\, \Theta(uv - m^2) \,. \tag{4.4.16}$$

The factor v decreases like m^2/u along branch "A" of the hyperbola. Thus in the low-energy region away from the origin,

$$(h_l)_u(\alpha_u) \;\sim\; m^2/E_P \qquad \text{whereas} \qquad \max_{v \in (0, E_P)} |(h_l)_u(v)| \;\sim\; E_P \,. \tag{4.4.17}$$

Therefore, we cannot assume that h_l satisfies the bound (4.3.23) with $n_0 = 0$. But $(h_l)^{(1)}_u(\alpha_u) \sim 1$ in the low-energy region, and thus we may choose $n_0 = 1$. We conclude that it is necessary to take into account two inner summands in (4.3.24), more precisely

$$\begin{aligned} P_l(s,l) \;&=\; \frac{1}{(il)^2} \sum_{k=0}^{\infty} \frac{(-il)^k}{k!} \int_{-\infty}^{\infty} \Big[(h_l)'_u(\alpha_u)\, \alpha_u^k \;-\; k\, (h_l)_u(\alpha_u)\, \alpha_u^{k-1} \Big]\, e^{-ius}\, du \\ &\quad + \cdots , \end{aligned} \tag{4.4.18}$$

where "$\cdots$" stands for the regularization expansion and all terms neglected in (4.3.24). In the low-energy region, we have according to (4.4.16, 4.3.29),

$$(h_l)_u(\alpha_u) \;=\; -\frac{1}{32\pi^3}\,\frac{m^2}{u} \;=\; (h_l)'_u(\alpha_u)\,\alpha_u\,.$$

Thus in this region, the two summands in the square brackets of (4.4.18) are of the same order of magnitude, and none of them can be neglected. In view of the low-energy limit, we introduce the regularization functions as

$$\begin{aligned} (h_l)'_u(\alpha_u) &=: -g_l(u) \\ (h_l)^{[1+n]}_u(\alpha_u) &=: -(n+1)\,g_l^{[n]}(u) \\ (h_l)'_u(\alpha_u)\,\alpha_u - (h_l)_u(\alpha_u) &=: \frac{b(u)}{u}\,g_l(u)\,; \end{aligned} \tag{4.4.19}$$

this yields the expansion (4.4.7). According to (4.4.5), the regularization functions have the low-energy limit (4.4.10). We finally consider the components $j = 2$ and 3. According to (4.4.5), these components are identically equal to zero in the low-energy limit. But for $u \sim E_P$, they might behave similar to P_s or P_l. To be on the safe side, we choose $n_0 = 1$. Denoting the regularization functions by

$$\begin{aligned} (h_{2\!/\!3})'_u(\alpha_u) &=: g_{2\!/\!3}(u) \\ (h_{2\!/\!3})^{[1+n]}_u(\alpha_u) &=: g^{[n]}_{2\!/\!3}(u) \\ -(h_{2\!/\!3})_u(\alpha_u) &=: \frac{b_{2\!/\!3}(u)}{u}\,g_{2\!/\!3}(u)\,, \end{aligned} \tag{4.4.20}$$

we obtain the expansion (4.4.8). According to (4.4.5), the regularization functions $g_{2\!/\!3}$, $g^{[h]}_{2\!/\!3}$ and $b_{2\!/\!3}$ vanish in the low-energy region, (4.4.11).

For clarity, we point out that choosing $n_0 = 1$ (as in (4.4.7, 4.4.8)) is a generalization of setting $n_0 = 0$ (as in (4.4.6)), obtained by taking into account more summands of the expansion (4.3.21). Nevertheless, the different behavior in the low-energy region (4.4.14, 4.4.17) suggests that (4.4.7) and (4.4.8) should not be merely more general formulas than (4.4.6), but that the behavior of $P_j(s,l)$, $j = l, 2, 3$, should be really different from that of $P_s(s,l)$. We shall now make this difference precise. Comparing (4.4.14) and (4.4.17) (and using that $h_{2\!/\!3}$ vanishes in the low-energy region), it is reasonable to impose that there should be a constant $\varepsilon_{\text{shear}} > 0$ with

$$|(h_j)_u(\alpha_u)| \;<\; \varepsilon_{\text{shear}}\,|(h_s)_u(\alpha_u)| \qquad \text{for } u \sim E_P \text{ and } j = l, 2, \text{ or } 3. \tag{4.4.21}$$

In view of (4.4.14) and (4.4.17), $\varepsilon_{\text{shear}}$ should be as small as

$$\varepsilon_{\text{shear}} \;\sim\; \frac{m^2}{E_P^2}\,. \tag{4.4.22}$$

However, if the surface states have shear (as defined earlier in this section), the constant $\varepsilon_{\text{shear}}$ must in general be chosen larger. In order to keep our analysis as general as possible, we will not specify here how $\varepsilon_{\text{shear}}$ scales in the Planck energy, but merely assume that $m^2/E_P^2 < \varepsilon_{\text{shear}} \ll 1$. Using (4.4.15), (4.4.19) and (4.4.20), the condition (4.4.21) can be expressed in terms of the regularization functions g_j and b_j as

$$\left(\frac{b}{u} + \alpha_u\right) g_l,\; \frac{b_{2\!/\!3}}{u}\,g_{2\!/\!3} \;<\; \varepsilon_{\text{shear}}\,u\,g_s \qquad \text{for } u \sim E_P. \tag{4.4.23}$$

It is interesting to discuss what the condition (4.4.21) means for the functions P_j. We begin with the case without regularization. In this case, the vector component of $P(x,y)$ points into the direction $y-x$, more precisely $P(x,y) = i(y-x)_j \gamma^j S(x,y)$ with a scalar distribution S. In a composite expression like the closed chain $P(x,y)\,P(y,x)$, one can contract the tensor indices and obtains in a formal calculation $P(x,y)P(y,x) = (y-x)^2\, S(x,y)\, S(y,x)$ with a scalar factor $(y-x)^2$ which vanishes on the light cone. Let us consider this contraction in our light-cone coordinates. Before the contraction, each factor $(y-x)_j \gamma^j = 2l\,\gamma^s + 2s\,\gamma^l \approx 2l\gamma^s$ is, if we take only the leading contribution on the light cone (i.e. the lowest order in s/l), proportional to l. After the contraction, however, the product $(y-x)^2 = 4ls$ is proportional to both l and s. Thus the contraction yields, to leading order on the light cone, a dimensionless factor s/l. While the factor l^{-1} changes the scaling behavior in the "large" variable, the factor s tends to make the composite expression "small" near the light cone. The analysis of the scaling behavior in l can immediately be extended to the case with regularization by looking at the expansions (4.4.6) and (4.4.7). Let us consider as an example the leading term of the mass expansion. For the expansion (4.4.6), this is the summand $k = 0$, and it scales like $P_s(s,l) \sim 1/l$. If we assume that (4.4.21) holds with $\varepsilon_{\text{shear}}$ according to (4.4.22), then (4.4.23) shows that $b(u) \sim 1$, and the summands in the square bracket in (4.4.7) are of comparable size. Hence the leading term of the expansion (4.4.7) is also the summand $k = 0$, and it scales in l like $P_l(s,l) \sim 1/l^2$. Hence the leading term of the sum $\gamma^l P_l + \gamma^s P_s$ behaves like $P \sim 1/l + \mathcal{O}(1/l^2)$. Since s and l are null directions, a contraction of the tensor indices in the closed chain leads only to mixed products of the form $P_s\, P_l$, and this mixed product scales in l like $P_s\, P_l \sim 1/l^3$. Thus, exactly as in the case without regularization, the contraction of the tensor indices yields an additional factor l^{-1}. If on the other hand, the condition (4.4.21) were violated, the regularization function b could be chosen arbitrarily large. But if b becomes large enough, the cominant contribution to (4.4.7) is the summand $k = 1$ (notice that b does not appear in the summand $k = 0$), and hence $P_l(s,l) \sim 1/l$. This implies that $P_s\, P_l \sim 1/l^2$, and the contraction does no longer yield an additional factor l^{-1}. This consideration is immediately extended to the components $P_{\not 2 3}$ by considering the l-dependence of the summands in (4.4.8). We conclude that the condition (4.4.21) with $\varepsilon_{\text{shear}} \ll 1$ means that the contraction of the tensor indices yields a scalar factor which is small on the light cone. We refer to this condition by saying that the *vector component is null on the light cone*. If one wishes, one can simply take this condition as an additional assumption on the fermionic projector of the vacuum. However, the property of the vector component being null on the light cone also arises in the study of composite expressions in the fermionic projector as a compatibility condition and can thus be derived from the equations of discrete space-time (see Remark 6.2.4)).

The next question is if our regularization functions α, g_j, $g_j^{[n]}$ and b, which appear in our expansions (4.4.6–4.4.8), are all independent of each other, or whether there are some relations between them. Recall that the regularization functions are given in terms of the boundary values of the functions $\partial_v^n h_j(u,v)$, $n \geq 0$, on the curve (u, α_u) (see (4.4.15, 4.4.19, 4.4.20)). Since the $(h_j)_{j=s,l,2,3}$ were treated in our two-dimensional Fourier analysis as four independent and (apart from our regularity assumptions) arbitrary functions, we can certainly not get relations between the regularization functions by looking at the situation in the uv-plane. But we can hope that when we consider the surface states in four-dimensional momentum space (as introduced in §4.3), the geometry of the hypersurface defined by these states might yield useful restrictions for

the regularization functions. First of all, we mention that our discussion of surface states of the previous section applies without changes also to the vector component; we will in what follows make use of the *restriction to surface states*. Since in the low-energy region the regularization is irrelevant and the results of §2.5 apply, we can furthermore restrict attention to large energy and momentum $\omega, |\vec{k}| \sim E_P$. We choose polar coordinates $(\omega, k = |\vec{k}|, \vartheta, \varphi)$ in momentum space and introduce the "mass shell coordinates"

$$U \;=\; -|\vec{k}| - \omega\,, \qquad\qquad V \;=\; |\vec{k}| - \omega\,. \tag{4.4.24}$$

Notice that, in contrast to the coordinates u and v, (4.3.6), the variables (4.4.24) are the spherically symmetric part of a four-dimensional coordinate system $(U, V, \vartheta, \varphi)$. Extending also the notation (4.4.4) in a spherically symmetric way, we introduce the Dirac matrix

$$\gamma^S \;=\; \frac{1}{2}\left(\gamma^0 \;-\; \frac{\vec{\gamma}\vec{k}}{k}\right).$$

Let us consider what the expansions (4.4.6–4.4.8) tell us about the surface states. Similar as explained before (4.3.30), the discontinuities of h_j come about in (4.4.2) when the plane $(u, v) = \text{const}$ meets and is tangential to the hypersurface of the surface states. We denote the tangential intersection point of the surface $(u, v) = \text{const}$ with the hypersurface by $Q = (U, V, \vartheta, \varphi)$. In the high-energy region under consideration, the variable U is of the order E_P. The variable V, on the other hand, will be of order $\alpha(U) < \alpha_{\max}$. Thus our hypersurface is close to the mass cone in the sense that $V/U \sim \alpha_{\max}/E_P \ll 1$. As a consequence, the coordinate ϑ of the intersection point Q must be small (more precisely, $\vartheta \leq \sqrt{\alpha_{\max}/E_P}$), and we conclude that, to leading order in $\alpha_{\max}/E_P$, $V = \alpha(U)$. Hence we can write the hypersurface as a graph $V = A(U, \vartheta, \varphi)$ with a function A satisfying the condition

$$A(U, \vartheta = 0) \;=\; \alpha(U) \;+\; (\text{higher orders in } \alpha_{\max}/E_P)\,.$$

One can think of the function $A(u, \vartheta, \varphi)$ as the extension of α to the four-dimensional setting. In order to determine the structure of the Dirac matrices, we first recall that the assumption that the vector component is null on the light cone implied in our consideration after (4.4.6) that the parameter n_0 corresponding to P_l, P_2 and P_3 was equal to one. This means that to leading order in $\alpha_{\max}/E_P$, only the function $h_s(u, v)$ is discontinuous on the curve (u, α_u), and we conclude that the distribution $\hat{P}$ is on the hypersurface at the point Q a scalar multiple of γ^s; we use the short notation $\hat{P}(Q) \sim \gamma^s$. Using again that ϑ is small, we obtain that to leading order in $\alpha_{\max}/E_P$, $\hat{P}(U, A(U, \vartheta = 0), \vartheta = 0) \sim \gamma^s$. Since the spatial direction of the vector $y - x$ in (4.1.3) can be chosen arbitrarily, we can by rotating our coordinate system immediately extend this result to general ϑ and φ, and obtain that $\hat{P}(U, \alpha(U, \vartheta, \varphi), \vartheta, \varphi) \sim \gamma^S$. Hence the surface states are described by a contribution to $\hat{P}$ of the form

$$-32\pi^3\, g(U, \vartheta, \varphi)\, \gamma^S\, \delta(V - A(U, \vartheta, \varphi)) \;+\; (\text{higher orders in } \alpha_{\max}/E_P) \tag{4.4.25}$$

with some function g. It is reasonable to assume that the functions in (4.4.25) are sufficiently regular. Similar to our regularity condition (4.3.12) for h, we here assume that the derivatives of A and g_S have the natural scaling behavior in E_P. More precisely, for all $n_1, n_2, n_3 \geq 0$ there should exist a constant $c \ll l E_P$ with

$$|\partial_U^{n_1} \partial_\vartheta^{n_2} \partial_\varphi^{n_3} A(U, \vartheta, \varphi)| \;+\; |\partial_U^{n_1} \partial_\vartheta^{n_2} \partial_\varphi^{n_3} g(U, \vartheta, \varphi)| \;\leq\; c\, E_P^{-n_1}\, \max(|A| + |g|) \tag{4.4.26}$$

for all $U \sim E_P$.

The form of the surface states (4.4.25) allows us to calculate the regularization functions g_j, $g_j^{[n]}$ and b_j. For this, we first represent the matrix γ^S in (4.4.25) in the Dirac basis $(\gamma^j)_{j=s,l,2,3}$; this yields the contributions of the surface states to the distributions $(v_j\, f)$. By substituting into (4.4.2) and carrying out the integrals over p_2 and p_3, one obtains the functions h_j. Finally, the regularization functions can be computed via (4.4.15, 4.4.19, 4.4.20). This whole calculation is quite straightforward, and we only state the main results. To leading order in v/u, we can take A and g as constant functions, and thus the calculation of $\gamma^s h_s + \gamma^l h_l$ reduces to the integral

$$-\frac{1}{\pi} \int_{-\infty}^{\infty} dp_2 \int_{-\infty}^{\infty} dp_3 \; \left(\gamma^s + \frac{v}{u}\, \gamma^l\right) g(u, \vartheta=0)\, \delta\Big(v - \alpha_u - \frac{p_2^2+p_3^2}{u}\Big)$$
$$+(\text{higher orders in } v/u,\, \alpha_{\max}/E_P)\,.$$

An evaluation in cylindrical coordinates yields that both $g_s(u)$ and $g_l(u)$ are equal to $g(u, \vartheta=0)$, and we thus have the important relation

$$g_s(u) \;=\; g_l(u) \;=:\; g(u)\,. \tag{4.4.27}$$

In the case without shear of the surface states, this relation was already found in (4.4.13); we now see that it holds in a much more general setting. The calculation of the angular components $j=2,3$ gives for $g_{2\!/\!3}$ contributions proportional to $u\,\partial_{2\!/\!3} A$ and $u\,\partial_{2\!/\!3} g$. Unfortunately, this is not very helpful because we have no information on the derivatives of A and g. The computation of the regularization functions $g_j^{[n]}$ involves higher derivatives of the functions in (4.4.25) and becomes quite complicated. We remark that the above analysis of the surface states can be carried out similarly for the scalar component of the previous section and gives relations between the regularization functions h and $h^{[n]}$, (4.3.25), but these relations all depend on unknown details of the geometry of the hypersurface. We thus conclude that (4.4.27) is the only relation between the regularization functions which can be derived with our present knowledge on the surface states,

We finally mention two assumptions on the regularization which, although we will not use them in the present work, might be worth considering later. The first assumption is related to the fact that P should as a projector be idempotent, $P^2 = P$. A formal calculation using (4.1.3) and (4.1.5) yields that

$$(P^2)(x,y) \;=\; \int \frac{d^4p}{(2\pi)^4}\, \hat{P}(p)^2\, e^{-ip(x-y)} \qquad \text{with} \tag{4.4.28}$$

$$\hat{P}(p)^2 \;=\; \big(2\phi(p)\, v_j(p)\, \gamma^j \;+\; (v_j(p)\, v^j(p) + \phi(p)^2)\big)\, f(p)^2\,. \tag{4.4.29}$$

In order to make sense out of (4.4.29), one must regularize in momentum space, e.g. by considering the system in finite 3-volume and take a suitable limit. Since the results of this analysis depend sensitively on how the regularization in momentum space is carried out, (4.4.29) cannot give any detailed information on the functions ϕ, v, or f. The only simple conclusion independent of the regularization is that the scalars multiplying the factors $v_j \gamma^j$ in (4.1.5) and (4.4.29) should have the same sign, and thus $\phi(p)\, f(p)$ should be positive. According to (4.3.4), this implies that the regularization function h be positive,

$$h(u) \;\geq\; 0 \qquad \text{for all } u.$$

This assumption is called the *positivity of the scalar component*. The second assumption is obtained by considering the rank of $\hat{P}(p)$. The 4×4 matrix $(\not{p} + m)$ in the

integrand of the unregularized fermionic projector (4.1.1) has the special property of being singular of rank two. This means that the fermionic projector is composed of only two occupied fermionic states, for every momentum p on the mass shell. The natural extension of this property to the case with regularization is that for every p on the hypersurface defined by the surface states, the matrix $\hat{P}(p)$ corresponding to the vector-scalar structure (4.1.5) should be of rank two. We refer to this property as the assumption of *half occupied surface states*. In terms of the functions $h(u,v)$ and $h_j(u,v)$, it means that $h_s(u,\alpha(u))\, h_l(u,\alpha_u) = h(u,\alpha_u)^2$. Using (4.3.25, 4.4.15, 4.4.19, 4.4.27), the assumption of half occupied surface states yields the following relation between the regularization functions of the scalar and vector components,

$$(\alpha(u)\, u + b(u))\, g(u)^2 \;=\; h(u)^2 \,. \tag{4.4.30}$$

4.5. The General Formalism

In this section we shall extend our previous analysis on the light cone in three ways: to the case with interaction, to systems of Dirac seas as introduced in §2.3 and to composite expressions in the fermionic projector. Our first step is to develop a method which allows us to introduce a regularization into the formulas of the light-cone expansion (2.5.45). We here only motivate and describe this method, the rigorous justification is given in Appendix D. Since the formulas of the light-cone expansion involve the factors $T^{(n)}$, (2.5.43, 2.5.42, 2.5.40), we begin by bringing these distributions into a form similar to our expansion of the regularized scalar component (4.3.26). By partly carrying out the Fourier integral (2.5.40) in the light-cone coordinates introduced in §4.3 (see (4.3.5, 4.3.6)), we can write the distribution T_a as

$$T_a(s,l) \;=\; \frac{1}{32\pi^3}\, \frac{1}{il} \int_0^\infty e^{-\frac{ial}{u} - ius}\, du \,. \tag{4.5.1}$$

This formula can be regarded as a special case of the expansion (4.3.20) (notice that the function $h(u,v)$ corresponding to T_a is computed similar to (4.3.8)), but (4.5.1) holds also away from the light cone. The distribution T_a is not differentiable in a at $a=0$, as one sees either directly in position space (2.5.41) or equivalently in (4.5.1), where formal differentiation leads to a singularity of the integrand at $u=0$. We bypassed this problem by working instead of T_a with the distribution T_a^{reg} (2.5.42). Let us briefly consider what this "regularization" means in the integral representation (4.5.1). The formal a-derivative of (4.5.1),

$$\frac{d}{da} T_a(s,l) \;=\; -\frac{1}{32\pi^3} \int_0^\infty \frac{1}{u}\, e^{-\frac{ial}{u} - ius}\, du \,,$$

is well-defined and finite for $a \neq 0$ because of the oscillatory factor $\exp(-ial/u)$. However, the limit $a \to 0$ leads to a logarithmic divergence. Thus one must subtract a logarithmic counterterm before taking the limit; more precisely,

$$T^{(1)}(s,l) \;=\; -\frac{1}{32\pi^3} \lim_{a\to 0} \int_{-\infty}^\infty \left[\frac{1}{u}\, e^{-\frac{ial}{u}}\, \Theta(u) \;-\; (1+\log a)\, \delta(u) \right] e^{-ius}\, du \,.$$

The higher a-derivatives $T^{(n)}$, $n>1$, are defined similarly using suitable counterterms which are localized at $u=0$. Since we do not need the details in what follows, we simply write

$$T^{(n)}(s,l) \;=\; -\frac{1}{32\pi^3}\, (-il)^{n-1} \int_0^\infty \left(\frac{1}{u^n} \right)^{\text{reg}} e^{-ius}\, du \,. \tag{4.5.2}$$

Consider a summand of the light-cone expansion (2.5.45),

$$\text{(phase-inserted line integrals) } T^{(n)}(s,l)\,. \tag{4.5.3}$$

According to our assumption of macroscopic potentials and wave functions described in §4.1, we shall regularize only the distribution $T^{(n)}$, keeping the iterated line integral unchanged. Let us briefly analyze what this assumption means quantitatively. Not regularizing the iterated line integral in (4.5.3), denoted in what follows by F, will be a good approximation if and only if F is nearly constant on the Planck scale. In other words, not regularizing F is admissible if we keep in mind that this method can describe the regularized fermionic projector only modulo contributions of the order $\partial_j F/E_P$. In the case that this last derivative acts on the bosonic potentials and fields contained in F, we obtain the limitation already mentioned in §4.1 that energy and momentum of the bosonic fields should be small compared to the Planck energy. More precisely, we can describe the fermionic projector only to leading order in $(l_{\text{macro}} E_P)^{-1}$, where l_{macro} is a typical length scale of macroscopic physics. A point we did not pay attention to earlier is that the iterated line integrals also involve factors $(y-x)$ which are contracted with the bosonic potentials and fields. Thus in light-cone coordinates, F will in general contain factors of l. If the derivative in $\partial_j F$ acts on a factor l, this factor is annihilated. Hence keeping the iterated line integrals in (4.5.3) unchanged can describe only the leading order in $(lE_P)^{-1}$ of the fermionic projector. We conclude that the assumption of macroscopic potentials and wave functions is justified if and only if we restrict our analysis to the *leading order in* $(lE_P)^{-1}$ *and* $(l_{macro} E_P)^{-1}$. We remark that going beyond the leading order in $(lE_P)^{-1}$ or $(l_{\text{macro}} E_P)^{-1}$ would make it impossible to describe the interaction by classical fields and is thus at present out of reach.

The restriction to the leading order in $(lE_P)^{-1}$ is a considerable simplification. First of all, we can neglect all regularization expansions (which are just expansions in powers of $(lE_P)^{-1}$; see e.g. (4.3.27) and the discussion thereafter), and thus we do not need to consider the regularization functions $h^{[n]}$ and $g_j^{[n]}$. Next we compare for given k the summands in (4.4.6–4.4.8) (the analysis for fixed k is justified assuming that the vector component is null on the light cone; see (4.4.21) and the discussion thereafter). One sees that the tensor index $j = s$ gives the leading contribution in $(lE_P)^{-1}$ to the vector component. This is a great simplification when tensor indices are contracted in composite expressions. Namely, when the vector component is contracted with the bosonic potentials or fields, it suffices to consider the contribution P_s, (4.4.6). If vector components are contracted with each other, the products of type $P_{2/3} P_{2/3}$ are according to (4.4.6–4.4.8) of higher order in $(lE_P)^{-1}$ or $\varepsilon_{\text{shear}}$ than corresponding products of type $P_s\, P_l$. Hence in such contractions, we must take into account both P_s and P_l, but we can again neglect the components P_2 and P_3. We conclude that the only regularization functions which should be of relevance here are those appearing in (4.3.26) and in the mass expansions of (4.4.6) and (4.4.7), i.e. the four functions

$$\alpha(u)\,, \qquad g(u)\,, \qquad h(u) \qquad \text{and} \qquad b(u) \tag{4.5.4}$$

with g given by (4.4.27).

Under the assumption of macroscopic potentials and wave functions, it suffices to regularize the factor $T^{(n)}$ in (4.5.3). Our method for regularizing $T^{(n)}$ is to go over to the integral representation (4.5.2) and to insert the regularization functions (4.5.4)

into the integrand. The procedure depends on whether the contribution to the light-cone expansion is of even or odd order in the mass parameter m. Furthermore, we must treat the factors $(y-x)_j \gamma^j$ in the light-cone expansion separately. The precise regularization method is the following.

Regularization of the light-cone expansion: *A summand of the light-cone expansion (2.5.45) which is proportional to m^p,*

$$m^p \,(\text{phase-inserted line integrals})\, T^{(n)}(s,l)\,, \tag{4.5.5}$$

has the regularization

$$\begin{aligned}
&(-1)\,(\text{phase-inserted line integrals}) \qquad (4.5.6)\\
&\times (-il)^{n-1} \int_{-\infty}^{\infty} du \left(\frac{1}{u^n}\right)^{\text{reg}} e^{-ius} \times \begin{cases} h(u)\, a(u)^{\frac{p-1}{2}} & \text{for } p \text{ odd} \\ g(u)\, a(u)^{\frac{p}{2}} & \text{for } p \text{ even} \end{cases}\\
&+ (\text{rapid decay in } l) \;+\; (\text{higher orders in } (lE_P)^{-1}, (l_{\text{macro}} E_P)^{-1}, \varepsilon_{\text{shear}})\,. \qquad (4.5.7)
\end{aligned}$$

A contribution to the light-cone expansion (2.5.45) which is proportional to m^p and contains a factor $(y-x)_j \gamma^j$,

$$m^p \,(\text{phase-inserted line integrals})\, (y-x)_j \gamma^j\, T^{(n)}(s,l)\,, \tag{4.5.8}$$

is properly regularized according to

$$\begin{aligned}
&(-1)\,(\text{phase-inserted line integrals})\\
&\times (-il)^{n-1} \int_{-\infty}^{\infty} du \left[2l\, \gamma^s \left(\frac{1}{u^n}\right)^{\text{reg}} + 2in\, \gamma^l \left(\frac{1}{u^{n+1}}\right)^{\text{reg}} + 2l\, b(u)\, \gamma^l \left(\frac{1}{u^{n+2}}\right)^{\text{reg}} \right]\\
&\times e^{-ius} \times \begin{cases} h(u)\, a(u)^{\frac{p-1}{2}} & \text{for } p \text{ odd} \\ g(u)\, a(u)^{\frac{p}{2}} & \text{for } p \text{ even} \end{cases} + (\text{contributions} \sim \gamma^2, \gamma^3)\\
&+ (\text{rapid decay in } l) \;+\; (\text{higher orders in } (lE_P)^{-1}, (l_{\text{macro}} E_P)^{-1}, \varepsilon_{\text{shear}})\,. \qquad (4.5.9)
\end{aligned}$$

In these formulas, the regularization function a is given by

$$a(u) \;=\; u\, \alpha(u)\,, \tag{4.5.10}$$

ε_{shear} is defined via (4.4.21), and l_{macro} is a macroscopic length scale.

Let us briefly explain and motivate this regularization method (see Appendix D for the derivation). First of all, we note that, after writing the factor $(y-x)_j \gamma^j$ together with the iterated line integrals, the expression (4.5.8) is of the form (4.5.5), and the regularization rule (4.5.7) applies. Thus (4.5.9) is an extension of (4.5.7) giving additional information on the l-component of the factor $(y-x)_j \gamma^j$. As we shall see later, this information is essential when the factor $(y-x)_j$ in (4.5.8) is to be contracted with another factor $(y-x)^j$ in a composite expression. To explain the formula (4.5.7), we first point out that the expansions of the scalar and vector components (4.3.26–4.3.28, 4.4.6, 4.4.7) do not involve the mass parameter m. The reason is that m was absorbed into the regularization functions g, h and α, as one sees by considering the low-energy limit; see (4.3.29, 4.4.9, 4.4.10). Furthermore, we note that each contribution to the mass expansions of the scalar or vector components contains either a factor h or g (see (4.3.26, 4.4.6, 4.4.7)), and it is therefore reasonable that we should also use exactly one of these factors here. As a consequence, the power m^p in (4.5.5) uniquely determines how many factors of each regularization function we should take. Namely for even p, we must take one factor g and $p/2$ factors α, whereas

the case of odd p gives rise to one factor h and $(p-1)/2$ factors α. On the other hand, we know that the insertion of the regularization functions into (4.5.2) should modify the behavior of the integrand only for large $u \sim E_P$; in particular, we should for small u have a behavior $\sim u^{-n}$. In order to comply with all these conditions, one must insert the regularization functions precisely as in (4.5.7). In order to motivate (4.5.9), we consider the expansion of the vector component (4.4.6, 4.4.7). Recall that the regularization function b vanishes in the low-energy region (4.4.10) and describes the shear of the surface states (as explained after (4.4.13)). Since this effect is not related to the mass of the Dirac particle, it is plausible that we should not associate to b a power of m. For the mass expansion of the vector component, we should thus collect all terms to a given power of α. The contribution $\sim \alpha^k$ to $\gamma^s P_s + \gamma^l P_l$ takes according to (4.4.6, 4.4.7) the form

$$\frac{1}{il}\,\frac{(-il)^k}{k!} \int_{-\infty}^{\infty} \left(-u\,\gamma^s \;+\; \frac{k-1}{il}\,\gamma^l \;-\; \frac{b}{u}\,\gamma^l \right) g\,\alpha^k\, e^{-ius}\, du\,.$$

In order to obtain the correct behavior in the low-energy region, we must multiply this formula by $-2l$ and choose $k = n+1$. This explains the form of the square bracket in (4.5.9). The combination of the regularization functions g, h and a in (4.5.9) can be understood exactly as in (4.5.7) using power counting in m.

Our constructions so far were carried out for the case $N = 1$ of one Dirac sea. We will now generalize our regularization method to systems of Dirac seas as introduced in (2.3) and will also introduce a compact notation for the regularization. Exactly as in §2.5 we only consider the *auxiliary fermionic projector*, because the fermionic projector is then obtained simply by taking the partial trace (2.3.20). We first outline how chiral particles (e.g. neutrinos) can be described. Without regularization, a chiral Dirac sea is obtained by multiplying the Dirac sea of massless particles with the chiral projectors $\chi_{L/R} = \frac{1}{2}(\mathbb{1} \mp \rho)$; for example in the vacuum and left/right handed particles,

$$\hat{P}(p) \;=\; \chi_{L/R}\, \not{p}\, \delta(p^2)\, \Theta(-p^0)\,. \tag{4.5.11}$$

The most obvious regularization method is to deduce the regularized chiral Dirac sea from a Dirac sea regularized with our above methods again by multiplying from the left with a chiral projector. This simple method indeed works, under the following assumptions. First, we must ensure that the regularized fermionic projector of the vacuum is a Hermitian operator. To this end, we must assume that the scalar component ϕ in (4.1.5) be identically equal to zero (this generalizes the requirement of massless particles needed in the case without regularization). Hence we regularize (4.5.11) by setting

$$\hat{P}(p) \;=\; \chi_{L/R}\, v_j(p) \gamma^j\, f(p)\,.$$

The expansions near the light cone are then obtained from (4.3.27, 4.3.28) and (4.4.6–4.4.8) by setting the scalar regularization functions h and $h^{[n]}$ to zero and by multiplying with $\chi_{L/R}$. Assuming furthermore that the bosonic potentials are causality compatible (see Def. 2.3.2), the formulas of the light-cone expansion are regularized likewise by taking the regularizations (4.5.7, 4.5.9) with h set identically equal to zero, and by multiplying from the left by a chiral projector $\chi_{L/R}$.

We next consider the generalization to systems of Dirac seas. In the vacuum, we can describe a system of Dirac seas by taking a direct sum of regularized Dirac seas and by using instead of the chiral projectors $\chi_{L/R}$ the chiral asymmetry matrix X (see (2.3.7)). Since we may choose the regularization functions for each Dirac

sea independently, this procedure clearly increases the total number of regularization functions. However, it is natural to impose that the regularization should respect all symmetries among the Dirac seas. More precisely, if the fermionic projector of the vacuum contains identical Dirac seas (e.g. corresponding to an underlying color $SU(3)$ symmetry), then we will always use the same regularization functions for all of these Dirac seas. Once the regularization has been specified for the vacuum, we can again apply the rules (4.5.5–4.5.9) to regularize the light-cone expansion. In the special case that the bosonic potentials are diagonal in the Dirac sea index, we can simply take the direct sum of the contributions (4.5.7, 4.5.9), using in each summand the regularization functions of the corresponding vacuum Dirac sea. In the general case of a non-diagonal bosonic field, the regularization functions can be inserted uniquely if one uses that, according to the assumption of macroscopic potentials and wave functions of §4.1, the fermionic projector is modified by the bosonic fields only on the macroscopic scale, so that its microscopic structure is the same as in the vacuum. For example, one can in the case of a gravitational and Yang-Mills field make the bosonic potential locally to zero by transforming to a suitable coordinate sytem and gauge, can in this system insert the regularization functions as in the vacuum and can finally transform back to the original system. We conclude that the generalization of our regularization method to systems of Dirac seas is quite straightforward and canonical. Therefore we can introduce a short notation for the regularizations of the factors $T^{(n)}$ in the light-cone expansion by simply adding a label for the order in the mass parameter. More precisely, we introduce in the case $N = 1$ of one Dirac sea the following abbreviations for the Fourier integrals in (4.5.7) and (4.5.9),

$$\begin{aligned} T^{(n)}_{[p]} &\equiv -(-il)^{n-1} \int_{-\infty}^{\infty} du \left(\frac{1}{u^n}\right)^{\text{reg}} e^{-ius} \\ &\qquad \times \begin{cases} h(u)\, a(u)^{\frac{p-1}{2}} & \text{for } p \text{ odd} \\ g(u)\, a(u)^{\frac{p}{2}} & \text{for } p \text{ even} \end{cases} \qquad (4.5.12) \\ (\not\xi\, T^{(n)}_{[p]}) &\equiv -(-il)^{n-1} \int_{-\infty}^{\infty} du\, e^{-ius} \times \begin{cases} h(u)\, a(u)^{\frac{p-1}{2}} & \text{for } p \text{ odd} \\ g(u)\, a(u)^{\frac{p}{2}} & \text{for } p \text{ even} \end{cases} \\ &\quad \times \left[2l\, \gamma^s \left(\frac{1}{u^n}\right)^{\text{reg}} + 2in\, \gamma^l \left(\frac{1}{u^{n+1}}\right)^{\text{reg}} + 2l\, b(u)\, \gamma^l \left(\frac{1}{u^{n+2}}\right)^{\text{reg}} \right] \qquad (4.5.13) \\ T^{(n)}_{\{p\}} &\equiv -(-il)^{n-1} \int_{-\infty}^{\infty} du \left(\frac{1}{u^n}\right)^{\text{reg}} e^{-ius}\, b(u) \\ &\qquad \times \begin{cases} h(u)\, a(u)^{\frac{p-1}{2}} & \text{for } p \text{ odd} \\ g(u)\, a(u)^{\frac{p}{2}} & \text{for } p \text{ even}\,. \end{cases} \qquad (4.5.14) \end{aligned}$$

In the case of a system of Dirac seas (2.3.3), we use the same notation for the corresponding direct sum. With this notation, the regularization of the light-cone expansion is carried out (modulo all the contributions neglected in (4.5.7) and (4.5.9)) merely by the replacement $m^p\, T^{(n)}(x,y) \to T^{(n)}_{[p]}$ and by marking with brackets that the factors $(y-x)_j \gamma^j$ and $T^{(n)}_{[p]}$ belong together (where we use the abbreviation $\xi \equiv y - x$) . We call a factor $\not\xi$ inside the brackets $(\not\xi T^{(n)}_{[p]})$ an *inner factor* $\not\xi$. Notice that the functions

$T^{(n)}_{\{p\}}$ in (4.5.14) involve the regularization function b; they will be needed below to handle contractions between the inner factors.

We finally come to the analysis of composite expressions in the fermionic projector. In §4.2 we already discussed the simplest composite expression, the closed chain $P(x,y)\,P(y,x)$ in the vacuum. In order to analyze the closed chain near the light cone, we substitute for $P(x,y)$ and $P(y,x)$ the regularized formulas of the light-cone expansion and multiply out. It is convenient to use that the fermionic projector is Hermitian and thus $P(y,x) = P(x,y)^*$; hence the light-cone expansion of $P(y,x)$ is obtained from that for $P(x,y)$ by taking the adjoint (with respect to the spin scalar product). The iterated line integrals can be multiplied with each other giving smooth functions; also we can simplify the resulting product of Dirac matrices using their anti-commutation relations. Denoting the adjoints of (4.5.12) and (4.5.13) by $\overline{T^{(n)}_{[p]}}$ and $(\overline{\not{\xi} T^{(n)}_{[p]}})$, respectively, we thus obtain for the closed chain a sum of terms of the following forms,

$$\begin{aligned} &F\, T^{(n_1)}_{[r_1]}\, \overline{T^{(n_2)}_{[r_2]}} \quad &,& \qquad F\, (\xi_{j_1} T^{(n_1)}_{[r_1]})\, \overline{T^{(n_2)}_{[r_2]}} \\ &F\, T^{(n_1)}_{[r_1]}\, \overline{(\xi_{j_2} T^{(n_2)}_{[r_2]})} \quad &,& \qquad F\, (\xi_{j_1} T^{(n_1)}_{[r_1]})\, \overline{(\xi_{j_2} T^{(n_2)}_{[r_2]})}\,, \end{aligned} \tag{4.5.15}$$

where F is a smooth function in x and y and n_j, r_j are integer parameters. Here the tensor indices of the inner factors ξ are contracted either with each other or with tensor indices in the smooth prefactor F. In order to analyze Euler-Lagrange equations like for example (3.5.20, 3.5.21), we need to consider more general expressions. More precisely, all Euler-Lagrange equations in this book can be written in terms of expressions being a product of a smooth function with a quotient of two monomials in $T^{(n)}_{[r]}$ and $\overline{T^{(n)}_{[r]}}$, possibly with inner factors ξ in the numerator. Thus our key problem is to mathematically handle expressions of the form

$$\begin{aligned} &(\text{smooth function}) \times \left[T^{(l_1)}_{[s_1]} \cdots T^{(l_f)}_{[s_f]}\, \overline{T^{(l_{f+1})}_{[s_{f+1}]} \cdots T^{(l_g)}_{[s_g]}} \right]^{-1} \\ &\quad \times (\xi_{j_1} T^{(n_1)}_{[r_1]}) \cdots (\xi_{j_a} T^{(n_a)}_{[r_a]})\, T^{(n_{a+1})}_{[r_{a+1}]} \cdots T^{(n_b)}_{[r_b]} \\ &\qquad \times \overline{(\xi_{j_{b+1}} T^{(n_{b+1})}_{[r_{b+1}]}) \cdots (\xi_{j_c} T^{(n_c)}_{[r_c]})\, T^{(n_{c+1})}_{[r_{c+1}]} \cdots T^{(n_d)}_{[r_d]}} \end{aligned} \tag{4.5.16}$$

with $0 \leq f \leq g$, $0 \leq a \leq b \leq c \leq d$, parameters l_j, s_i, n_i, p_i and tensor indices j_i. Here the tensor indices of the inner factors ξ_i are again contracted either with other inner factors or with tensor indices in the smooth prefactor. We mention for clarity that, since the factors in (4.5.16) are complex functions or, in the case $N > 1$ of systems of Dirac seas, direct sums of complex functions, the product (4.5.16) clearly is commutative.

The inner factors in (4.5.16) can be simplified using the particular form (4.5.12, 4.5.13) of $T^{(n)}_{[r]}$ and $(\xi_j T^{(n)}_{[r]})$. We begin with the case of an inner factor which is contracted with a tensor index in the smooth prefactor, i.e. with products of the form

$$\cdots F^j\, (\xi_j T^{(n)}_{[r]}) \cdots \qquad \text{or} \qquad \cdots F^j\, \overline{(\xi_j T^{(n)}_{[r]})} \cdots$$

and a smooth vector field F, where "$\cdots$" stands for any other factors of the form as in (4.5.16). According to (4.5.13), to leading order in $(lE_P)^{-1}$ it suffices to take into

account the s-component, and thus (4.5.12) yields that $(\not{\xi} T^{(n)}_{[r]}) \approx 2l\, \gamma^s\, T^{(n)}_{[r]}$. Since $2l\,\gamma^s$ coincides on the light cone with $\not{\xi}$, we conclude that, to leading order in $(lE_P)^{-1}$,

$$F^j\,(\xi_j T^{(n)}_{[r]}) \;=\; F^j \xi_j\, T^{(n)}_{[r]} \qquad \text{and} \qquad F^j\,(\overline{\xi_j\, T^{(n)}_{[r]}}) \;=\; F^j \xi_j \overline{T^{(n)}_{[r]}}\,. \tag{4.5.17}$$

These relations coincide with what one would have expected naively. We next consider the case of two inner factors which are contracted with each other, i.e. products of the following form,

$$\begin{aligned} &\cdots (\xi_j T^{(n_1)}_{[r_1]})(\xi^j T^{(n_2)}_{[r_2]}) \cdots \quad , \qquad \cdots (\xi_j T^{(n_1)}_{[r_1]})(\overline{\xi^j T^{(n_2)}_{[r_2]}}) \cdots \\ &\qquad\qquad \text{or} \qquad\qquad \cdots (\overline{\xi_j T^{(n_1)}_{[r_1]}})(\overline{\xi^j T^{(n_2)}_{[r_2]}}) \cdots\,. \end{aligned} \tag{4.5.18}$$

In this case, the product cannot be calculated naively because the factor $\xi_j \xi^j = \xi^2$ vanishes on the light cone. But we can still compute the product using the Fourier representation (4.5.13). Since the s- and l-directions are null, only the mixed products of the s- and l-components in (4.5.13) contribute, and we obtain

$$\begin{aligned} &(\xi_j T^{(n_1)}_{[r_1]})(\xi^j T^{(n_2)}_{[r_2]}) \\ &\quad= \quad (-il)^{n_1-1}\, l \int_{-\infty}^{\infty} du_1\, \frac{1}{u_1^{n_1}}\, e^{-iu_1 s} \times \begin{cases} h(u_1)\, a(u_1)^{\frac{r_1-1}{2}} & \text{for } r_1 \text{ odd} \\ g(u_1)\, a(u_1)^{\frac{r_1}{2}} & \text{for } r_1 \text{ even} \end{cases} \\ &\qquad \times (-il)^{n_2-1} \int_{-\infty}^{\infty} du_2 \left[\frac{2in_2}{u_2^{n_2+1}} + \frac{2l\, b(u_2)}{u_2^{n_2+2}} \right] e^{-iu_2 s} \\ &\qquad\qquad \times \begin{cases} h(u_2)\, a(u_2)^{\frac{r_2-1}{2}} & \text{for } r_2 \text{ odd} \\ g(u_2)\, a(u_2)^{\frac{r_2}{2}} & \text{for } r_2 \text{ even} \end{cases} \\ &\qquad + (-il)^{n_1-1} \int_{-\infty}^{\infty} du_1 \left[\frac{2in_1}{u_1^{n_1+1}} + \frac{2l\, b(u_1)}{u_1^{n_1+2}} \right] e^{-iu_1 s} \\ &\qquad\qquad \times \begin{cases} h(u_1)\, a(u_1)^{\frac{r_1-1}{2}} & \text{for } r_1 \text{ odd} \\ g(u_1)\, a(u_1)^{\frac{r_1}{2}} & \text{for } r_1 \text{ even} \end{cases} \\ &\qquad \times (-il)^{n_2-1}\, l \int_{-\infty}^{\infty} du_2\, \frac{1}{u_2^{n_2}}\, e^{-iu_2 s} \times \begin{cases} h(u_2)\, a(u_2)^{\frac{r_2-1}{2}} & \text{for } r_2 \text{ odd} \\ g(u_2)\, a(u_2)^{\frac{r_2}{2}} & \text{for } r_2 \text{ even}\,, \end{cases} \end{aligned}$$

and similarly for the two other products in (4.5.18). In the case of systems of Dirac seas, this calculation can be done for each summand of the direct sum separately. Rewriting the Fourier integrals using the notation (4.5.12) and (4.5.14), we get the following result.

Contraction rules: *To leading order in* $(lE_P)^{-1}$,

$$\begin{aligned} &(\xi_j T^{(n_1)}_{[r_1]})(\xi^j T^{(n_2)}_{[r_2]}) \\ &\quad= \quad -2\, T^{(n_1)}_{[r_1]}\, (n_2\, T^{(n_2+1)}_{[r_2]} + T^{(n_2+2)}_{\{r_2\}}) \;-\; 2\, (n_1\, T^{(n_1+1)}_{[r_1]} + T^{(n_1+2)}_{\{r_1\}})\, T^{(n_2)}_{[r_2]} \end{aligned} \tag{4.5.19}$$

$$(\xi_j T^{(n_1)}_{[r_1]})(\overline{\xi^j T^{(n_2)}_{[r_2]}})$$
$$= -2\, T^{(n_1)}_{[r_1]}\, (n_2\, \overline{T^{(n_2+1)}_{[r_2]}} + \overline{T^{(n_2+2)}_{\{r_2\}}}) \;-\; 2\, (n_1\, T^{(n_1+1)}_{[r_1]} + T^{(n_1+2)}_{\{r_1\}})\, \overline{T^{(n_2)}_{[r_2]}} \tag{4.5.20}$$
$$(\overline{\xi_j T^{(n_1)}_{[r_1]}})(\overline{\xi^j T^{(n_2)}_{[r_2]}})$$
$$= -2\, \overline{T^{(n_1)}_{[r_1]}}\, (n_2\, \overline{T^{(n_2+1)}_{[r_2]}} + \overline{T^{(n_2+2)}_{\{r_2\}}}) \;-\; 2\, (n_1\, \overline{T^{(n_1+1)}_{[r_1]}} + \overline{T^{(n_1+2)}_{\{r_1\}}})\, \overline{T^{(n_2)}_{[r_2]}}\,. \tag{4.5.21}$$

By iteratively applying (4.5.17) and the contraction rules (4.5.19–4.5.21), we can in (4.5.16) eliminate all inner factors ξ to end up with products of the form

$$(\text{smooth function})\, \frac{T^{(a_1)}_\circ \cdots T^{(a_\alpha)}_\circ\, \overline{T^{(b_1)}_\circ \cdots T^{(b_\beta)}_\circ}}{T^{(c_1)}_\circ \cdots T^{(c_\gamma)}_\circ\, \overline{T^{(d_1)}_\circ \cdots T^{(d_\delta)}_\circ}} \tag{4.5.22}$$

with parameters $\alpha, \beta \geq 1$, $\gamma, \delta \geq 0$ and a_i, b_i, c_i, d_i (if $\gamma = 0 = \delta$ the denominator clearly is equal to one). Here each subscript "$\circ$" stands for an index $[r]$ or $\{r\}$. The quotient of the two monomials in (4.5.22) is called a *simple fraction*.

We point out that the above transformation rules for the inner factors (4.5.17) and (4.5.19–4.5.21) are identities valid pointwise (i.e. for fixed x and y) close to the light cone. We anticipate that Euler-Lagrange equations like (3.5.20, 3.5.21) do not lead us to evaluate the products of the form (4.5.16) pointwise, but merely in the weak sense. Therefore, we now go over to a weak analysis of the simple fraction. In the case of a continuous regularization, we thus consider the integral

$$\int d^4x\, \eta(x)\, \frac{T^{(a_1)}_\circ \cdots T^{(a_\alpha)}_\circ\, \overline{T^{(b_1)}_\circ \cdots T^{(b_\beta)}_\circ}}{T^{(c_1)}_\circ \cdots T^{(c_\gamma)}_\circ\, \overline{T^{(d_1)}_\circ \cdots T^{(d_\delta)}_\circ}} \tag{4.5.23}$$

with a test function η. Before coming to the derivation of calculation rules for the integrand in (4.5.23), we must think about how the test function η is to be chosen. As explained in §4.2 in the example of the closed chain (4.2.2), a weak integral in general depends essentially on the unknown high-energy behavior of the fermionic projector and is therefore undetermined. To avoid this problem, we must evaluate (4.5.23) in such a way that our expansions near the light cone become applicable. To this end, we assume that η has its *support near the light cone*, meaning that in light-cone coordinates (s, l, x_2, x_3), the "large" variable l satisfies on the support of η the conditions (4.3.10). For clarity, we remark that this definition does not state that the support of η should be in a small neighborhood of the light cone, but merely in a strip away from the origin. This is sufficient because we shall extract information on the behavior near the light cone by considering the singularities of the integral for $E_P \to \infty$ (see (4.5.29) below). Furthermore, we assume that η is *macroscopic* in the sense that its partial derivatives scale in powers of l^{-1} or l^{-1}_{macro}. Under these assumptions, the integrand in (4.5.23) is macroscopic in l, and carrying out the s- and l-integrals in (4.5.23) gives a function which is macroscopic in the "transversal" variables x_2 and x_3. Therefore, in the three variables (l, x_2, x_3), a weak analysis is equivalent to a pointwise analysis, and thus it suffices to consider the s-integral in (4.5.23), i.e. the expression

$$\int_{-\infty}^{\infty} ds\, \eta\, \frac{T^{(a_1)}_\circ \cdots T^{(a_\alpha)}_\circ\, \overline{T^{(b_1)}_\circ \cdots T^{(b_\beta)}_\circ}}{T^{(c_1)}_\circ \cdots T^{(c_\gamma)}_\circ\, \overline{T^{(d_1)}_\circ \cdots T^{(d_\delta)}_\circ}} \tag{4.5.24}$$

for fixed l, x_2 and x_3. In the case of a discrete regularization, the integral in (4.5.23) must be replaced by a sum over all space-time points, i.e. we must consider instead of (4.5.23) the weak sum

$$\sum_{x \in M} \eta(x)\, \frac{T_\circ^{(a_1)} \cdots T_\circ^{(a_\alpha)}\, \overline{T_\circ^{(b_1)} \cdots T_\circ^{(b_\beta)}}}{T_\circ^{(c_1)} \cdots T_\circ^{(c_\gamma)}\, \overline{T_\circ^{(d_1)} \cdots T_\circ^{(d_\delta)}}}\,, \tag{4.5.25}$$

where $M \subset \mathbb{R}^4$ are the discrete space-time points and η is a macroscopic function in $\mathbb{R}^4$ with support near the light cone. Up to a normalization factor, (4.5.25) can be regarded as a Riemann sum which approximates the integral (4.5.23). Assuming that the space-time points have a generic position in $\mathbb{R}^4$ and keeping in mind that the function inside the sum (4.5.25) is macroscopic in the variables l, x_2, and x_3, the Riemann sum and the integral indeed coincide to leading order in $(lE_P)^{-1}$ and $(l_{\text{macro}} E_P)^{-1}$. Hence it is admissible to work also in the discrete case with the one-dimensional integral (4.5.24).

Let us analyze the integral (4.5.24) in more detail. We first consider how (4.5.24) scales in the Planck energy. In the limit $E_P \to \infty$, the factors $T_\circ^{(n)}$ go over to distributions which are in general singular on the light cone. Hence their product in (4.5.24) becomes ill-defined for $E_P \to \infty$ even in the distributional sense, and thus we expect that the integral (4.5.24) should diverge for $E_P \to \infty$. The order of this divergence can be determined using the following power counting argument. Keeping in mind that the regularization functions decay on the Planck scale $u \sim E_P$, the Fourier integrals (4.5.12) and (4.5.14) behave on the light cone (i.e. for $s = 0$) like

$$T_\circ^{(n)} \;\sim\; \log^g(E_P)\, E_P^{-n+1}$$

with $g = 1$ in the case $n = 1$ and $g = 0$ otherwise. Hence the product in the integrand of (4.5.24) scales on the light cone as

$$\frac{T_\circ^{(a_1)} \cdots T_\circ^{(a_\alpha)}\, \overline{T_\circ^{(b_1)} \cdots T_\circ^{(b_\beta)}}}{T_\circ^{(c_1)} \cdots T_\circ^{(c_\gamma)}\, \overline{T_\circ^{(d_1)} \cdots T_\circ^{(d_\delta)}}} \;\sim\; \log^g(E_P)\, E_P^L \tag{4.5.26}$$

with $g \in \mathbb{Z}$ and

$$L \;=\; \alpha + \beta - \gamma - \delta - \sum_{j=1}^{\alpha} a_j - \sum_{j=1}^{\beta} b_j + \sum_{j=1}^{\gamma} c_j + \sum_{j=1}^{\delta} d_j\,. \tag{4.5.27}$$

We call L the *degree* of the simple fraction. We will here restrict attention to the case $L > 1$. In this case, the simple fraction (4.5.26) diverges in the limit $E_P \to \infty$ at least quadratically. If s is not zero, the oscillations of the factor $\exp(-ius)$ in (4.5.12, 4.5.14) lead to a decay of $T_\circ^{(n)}$ on the scale $s \sim E_P^{-1}$. This consideration shows that the dominant contribution to the integral (4.5.24) when $E_P \to \infty$ is obtained by evaluating η on the light cone, and the scaling behavior of this contribution is computed by multiplying (4.5.26) with a factor E_P^{-1}. We conclude that (4.5.24) diverges in the limit $E_P \to \infty$, and its leading divergence scales in E_p like

$$\eta(s=0)\, \log^g(E_P)\, E_P^{L-1}\,. \tag{4.5.28}$$

Collecting the logarithmic terms in the light-cone expansion, one can easily compute the parameter g. We remark that due to possible zeros in the denominator, the integral (4.5.24) might diverge even for finite E_P. In this case we can still use (4.5.28) if we set the proportionality factor equal to plus or minus infinity. We also note that, by

substituting the Fourier representations (4.5.12, 4.5.14) into (4.5.24), one can rewrite the products in (4.5.24) in terms of the regularization functions (this is explained in detail in Appendix E for a particular choice of regularization functions). Collecting the factors of l in (4.5.12) and (4.5.14), we end up with the following result.

Weak evaluation near the light cone: *Consider the integral (4.5.24) for a simple fraction of degree $L > 1$. Then there is an integer $g \geq 0$ and a real coefficient c_{reg} independent of s and l such that for every macroscopic test function η,*

$$\int_{-\infty}^{\infty} ds\, \eta\, \frac{T_\circ^{(a_1)} \cdots T_\circ^{(a_\alpha)}\, \overline{T_\circ^{(b_1)} \cdots T_\circ^{(b_\beta)}}}{T_\circ^{(c_1)} \cdots T_\circ^{(c_\gamma)}\, \overline{T_\circ^{(d_1)} \cdots T_\circ^{(d_\delta)}}} \;=\; \frac{c_{\text{reg}}}{(il)^L}\, \eta(s=0)\, \log^g(E_P)\, E_P^{L-1} + (\text{higher orders in } (lE_P)^{-1} \text{ and } (l_{\text{macro}} E_P)^{-1})\,. \tag{4.5.29}$$

The coefficient c_{reg} clearly depends on the indices of the simple fraction and on the details of the regularization. We call c_{reg} a *regularization parameter*.

Integrals of type (4.5.24) can be transformed using integration by parts. For clarity we begin with the special case of a monomial,

$$\int_{-\infty}^{\infty} ds\, \left(\frac{d}{ds}\eta\right) T_\circ^{(a_1)} \cdots \overline{T_\circ^{(b_q)}} \;=\; -\int_{-\infty}^{\infty} ds\, \eta\, \frac{d}{ds}\left(T_\circ^{(a_1)} \cdots \overline{T_\circ^{(b_q)}}\right) \tag{4.5.30}$$

$$= \;-\int_{-\infty}^{\infty} ds\, \eta \left[\left(\frac{d}{ds} T_\circ^{(a_1)}\right) T_\circ^{(a_2)} \cdots \overline{T_\circ^{(b_q)}} + \cdots + T_\circ^{(a_1)} \cdots \overline{T_\circ^{(b_{q-1})}} \left(\frac{d}{ds} \overline{T_\circ^{(b_q)}}\right)\right], \tag{4.5.31}$$

where in the last step we applied the Leibniz rule. Differentiating (4.5.12) and (4.5.14) with respect to s yields that

$$\frac{d}{ds} T_\circ^{(n)} \;=\; -l\, T_\circ^{(n-1)} \qquad \text{and} \qquad \frac{d}{ds} \overline{T_\circ^{(n)}} \;=\; -l\, \overline{T_\circ^{(n-1)}}\,. \tag{4.5.32}$$

With these relations, we can carry out the derivatives in (4.5.31). Notice that the differentiation rules (4.5.32) decrease the index n by one. According to (4.5.27) and (4.5.29), decrementing the upper index of a factor $T_\circ^{(a_j)}$ or $\overline{T_\circ^{(b_k)}}$ increments the degree of the monomial and yields in the weak integral a factor of the order E_P/l. Using furthermore that η is macroscopic (as defined after (4.5.23)), we conclude that each summand in (4.5.31) dominates the left side of (4.5.30) by one order in lE_P or $l_{\text{macro}} E_P$. We have thus derived the following result.

Integration-by-parts rule for monomials: *Consider a monomial of degree $L > 1$. In a weak analysis near the light cone, we have to leading order in $(lE_P)^{-1}$ and $(l_{macro} E_P)^{-1}$,*

$$0 \;=\; T_\circ^{(a_1-1)} \cdots T_\circ^{(a_p)}\, \overline{T_\circ^{(b_1)}} \cdots \overline{T_\circ^{(b_q)}} + \cdots + T_\circ^{(a_1)} \cdots T_\circ^{(a_p-1)}\, \overline{T_\circ^{(b_1)}} \cdots \overline{T_\circ^{(b_q)}} + T_\circ^{(a_1)} \cdots T_\circ^{(a_p)}\, \overline{T_\circ^{(b_1-1)}} \cdots \overline{T_\circ^{(b_q)}} + \cdots + T_\circ^{(a_1)} \cdots T_\circ^{(a_p)}\, \overline{T_\circ^{(b_1)}} \cdots \overline{T_\circ^{(b_q-1)}}\,. \tag{4.5.33}$$

The integration-by-parts method works similarly for simple fractions. For ease in notation we state it more symbolically.

Integration-by-parts rule for simple fractions: *Consider a simple fraction of*

degree $L > 1$. In a weak analysis near the light cone and to leading order in $(lE_P)^{-1}$ and $(l_{macro}E_P)^{-1}$,

$$\nabla \left(\frac{T_\circ^{(a_1)} \cdots T_\circ^{(a_\alpha)}\, \overline{T_\circ^{(b_1)} \cdots T_\circ^{(b_\beta)}}}{T_\circ^{(c_1)} \cdots T_\circ^{(c_\gamma)}\, \overline{T_\circ^{(d_1)} \cdots T_\circ^{(d_\delta)}}} \right) \;=\; 0\,. \tag{4.5.34}$$

Here ∇ acts on all factors like a derivation (i.e. it is linear and satisfies the Leibniz rule), commutes with complex conjugations and

$$\nabla T_\circ^{(n)} \;=\; T_\circ^{(n-1)}\,, \qquad \nabla \frac{1}{T_\circ^{(n)}} \;=\; -\frac{T_\circ^{(n-1)}}{(T_\circ^{(n)})^2}\,.$$

The integration-by-parts rule gives us relations between simple fractions. We say that simple fractions are *independent* if the integration-by-parts rules gives no relations between them. More systematically, we consider the vector space of linear combinations of simple fractions. We say that two such linear combinations are *equivalent* if they can be transformed into each other with the integration-by-parts rules. We refer to the equivalence classes as the *basic fractions*. Taking the linear combination of the corresponding regularization parameters c_{reg}, we can associate to every basic fraction a so-called *basic regularization parameter*. In Appendix E it is shown for all simple fractions which will appear in this book that the corresponding basic fractions are linearly independent in the sense that there are no further identities between them. Therefore it seems a reasonable method to take the basic regularization parameters as empirical parameters modeling the unknown microscopic structure of space-time.

We remark that the notion of the basic fraction can be made more concrete by choosing from each equivalence class one representative. Then one can identify every basic fraction with the distinguished simple fraction in its equivalence class. For simplicity we give this construction in the special case that the simple fractions are monomials of the form

$$T_\circ^{(a_1)} \cdots T_\circ^{(a_p)}\, \overline{T_\circ^{(b_1)}} \cdots \overline{T_\circ^{(b_q)}}$$

(the construction can immediately be extended to simple fractions, but it becomes a bit complicated and we do not need it here). If only one factor $T_\circ^{(a)}$ appears ($p = 1$), one can by applying the integration-by-parts rule iteratively increment the parameter a_1; this clearly decreases the other parameters $b_1, \ldots, b_q$. In order to avoid that any of the parameters $b_1, \ldots, b_q$ becomes smaller than -1, we stop the integration-by-parts procedure as soon as one of the b_j equals -1. In this way, we can express every monomial as a unique linear combination of monomials of the form

$$T_\circ^{(a_1)}\, \overline{T_\circ^{(b_1)}} \cdots \overline{T_\circ^{(b_q)}} \qquad \text{with} \qquad -1 = b_1 \leq \cdots \leq b_q. \tag{4.5.35}$$

Similarly for $p > 1$, the integration-by-parts rule allows us to increment the smallest of the parameters a_j, unless either one of the parameters b_j equals -1 or there are two factors $T^{(a_j)}$ with $a_j = \min(a_1, \ldots, a_p)$. By iteration, we can thus transform any monomial into a linear combination of monomials of the following type,

$$\begin{aligned} T_\circ^{(a_1)} \cdots T_\circ^{(a_p)}\, \overline{T_\circ^{(b_1)}} \cdots \overline{T_\circ^{(b_q)}} \quad & \text{with} \quad a_1 \leq \cdots \leq a_p,\;\; b_1 \leq \cdots \leq b_q \\ & \text{and} \quad a_1 = a_2 \text{ or } b_1 = -1\,. \end{aligned} \tag{4.5.36}$$

We can now consider (4.5.35) and (4.5.36) as the basic monomials.

With the above constructions we have developed the mathematical framework for analyzing composite expressions in the fermionic projector in the continuum. Our procedure is outlined as follows. We first substitute for the fermionic projector the regularized formulas of the light-cone expansion; this yields sums of products of the form (4.5.16), where the smooth prefactor involves the bosonic potentials and fields as well as the wave functions of the Dirac particles and anti-particles of the system. Applying our contraction rules, we then eliminate all inner factors and obtain terms of the form (4.5.22). When evaluated in the weak sense (4.5.29), the l-dependence determines the degree L of the simple fraction, and the dependence on the regularization is described for each simple fraction by the corresponding regularization parameters c_{reg}. Using our integration-by-parts rule, we can furthermore restrict attention to the basic fractions and the corresponding basic regularization parameters. Taking the basic regularization parameters as free empirical parameters, the composite expressions in the fermionic projector reduce to expressions in the bosonic fields and fermionic wave functions, involving a small number of free parameters. This procedure for analyzing composite expressions in the fermionic projector is called the *continuum limit*.

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